# ON THE CLASSIFICATION OF PRODUCT-QUOTIENT SURFACES WITH $q = 0$, $p_g = 3$ AND THEIR CANONICAL MAP

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Abstract. In this work we present new results to produce an algorithm that returns, for any fixed pair of natural integers $K^2$ and $\chi$, all regular surfaces $S$ of general type with self-intersection $K_S^2 = K^2$ and Euler characteristic $\chi(\mathcal{O}_S) = \chi$, that are product-quotient surfaces.
The key result we obtain is an algebraic characterization of all families of regular product-quotients surfaces, up to isomorphism, arising from a pair of $G$-coverings of $\mathbb{P}^1$.
As a consequence of our work, we provide a classification of all regular product-quotient surfaces of general type with $23 \leq K^2 \leq 32$ and $\chi(\mathcal{O}_S) = 4$. Furthermore, we study their canonical map and present several new examples of surfaces of general type with a high degree of the canonical map.

Contents



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## Introduction

The history of the canonical map of surfaces of general type is long more than 45 years and it has been recently revived after the beautiful survey [28], where the authors provide an overview of the current state of knowledge on the topic, also outlining a series of still-open questions.

In 1978, Persson proved that the degree of the canonical map of surfaces of general type is bounded from above by 36, see [30]. Furthermore, it is known since [9] that if the degree is more than 27, then $q = 0$ and $p_g = 3$.
For a long time, the only examples with a high degree of the canonical map were the surfaces of Persson [30] with degree 16 and Tan [35], with degree 12, proving how much challenging can be the construction of new examples. Recently it has been proved that the bound given by Persson is sharp, see [34], [27], [32]. As a consequence of this, M. Mendes Lopes and R. Pardini revived the topic of the degree of the canonical map and posed in their survey, among other things, the natural question [28, Question 5.2] if all the integers between 2 and 36 can be the degree of the canonical map of some surfaces of general type having $q = 0$ and $p_g = 3$.
We also noteworthy, as mentioned in [9], that the degree of the canonical map is bounded from above by $K_S^2$, so that minimal surfaces with a high degree of the canonical map not only have $q = 0$ and $p_g = 3$ but also high values of $K_S^2$.

In this paper, we construct product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $23 \leq K_S^2 \leq 32$ with the ultimate goal to compute the degree of their canonical map and give new examples.
We remind that $K_S^2 = 32$ is the highest possible value for product-quotient surfaces with $q = 0$ and $p_g = 3$, see Theorem 2.3.

We consider product-quotient surfaces as they have proven to be highly useful tools in investigating unresolved conjectures in Algebraic Geometry. As a series of examples, that only deal with regular surfaces, we mention the rigid but not infinitesimally rigid manifolds [6] constructed by Bauer and Pignatelli that gave a negative answer to a question of Kodaira and Morrow [29, p. 45], the families of surfaces with $p_g = q = 0$ constructed in [8] realizing 13 new topological types and for which Bloch's conjecture [12] holds, and the series of papers [7], [8], [3], [2], [1] providing a classification of minimal product-quotient surfaces of general type with $p_g = q = 0$ to give a partial answer to a still-open problem posed by Mumford in 1980, see [2, p. 551].

As a first result of this paper, we refine the MAGMA code of [8] and we present a new version of it which, taking as input a pair of natural integers $K^2$ and $\chi$, returns all regular surfaces $S$ of general type with self-intersection $K_S^2 = K^2$ and Euler characteristic $\chi(\mathcal{O}_S) = \chi$, that are product-quotient surfaces.

Although the original script is relatively easy to be adapted to any fixed value of $\chi$ and not only for $\chi = 1$ as in [8], it still presents computational time problems as the value of $\chi$ increases. We make the code more performant by giving new improvements.

A first novelty is the implementation of the database and the script *FindGenerators* developed in [16]. Such database contains one spherical system of generators of a finite group $G$ for each family of pairwise topologically equivalent $G$-coverings $C$ of $\mathbb{P}^1$, where the genus of $C$ is $g(C) \leq 27$. We use these tools from [16] to speed up **Step 3** in Subsection 2.1 as well.
The second main novelty is given from the following new result:

**Theorem 0.1.** *Let $V_1, V_2$ be two spherical systems of generators of a finite group $G$. Assume that the associated topological types of $G$-coverings of $\mathbb{P}^1$ are different. The families of product-quotient surfaces associated to this pair of topological types of $G$-coverings is in natural bijection with the set of double cosets*

$$\mathcal{B}\operatorname{Aut}(G, V_1)\backslash \operatorname{Aut}(G)/\mathcal{B}\operatorname{Aut}(G, V_2).$$

This is a short version of the main Theorem 1.19. We also remind to the definitions in Subsection 1.2 that make clear the objects presented in Theorem 0.1. The analogous case of Theorem 0.1 where $V_1$ and $V_2$ have topological equivalent associated $G$-coverings of $\mathbb{P}^1$ is discussed in Corollary 1.21.

As a consequence of these improvements, we run the above-mentioned script to obtain a classification of regular product-quotient surfaces $S$ with $23 \leq K_S^2 \leq 32$ and $\chi(\mathcal{O}_S) = 4$. What we obtain is the following

**Theorem 0.2.** *Let $S$ be a regular product-quotient surface with $23 \leq K_S^2 \leq 32$ and $\chi(\mathcal{O}_S) = 4$. Then $S$ is a surface of general type and it realizes one of the families of surfaces described in tables 17 to 29 in the appendix of this paper. Moreover, surfaces in tables 17 to 28 are minimal.*

We are interested to compute the degree of the canonical map of product-quotient surfaces, with a particular focus to those with $p_g = 3$.
Let $S$ be a product-quotient surface given by a pair of curves $C_1$ and $C_2$ and a finite group $G$. We prove that the degree of the canonical map of $S$ is determined automatically whenever we compute the (schematic) base locus of the linear subsystem associated to the subspace $H^{2,0}(C_1 \times C_2)^G$ of invariant 2-forms of $C_1 \times C_2$.

Such subspace splits as a direct sum of subspaces $\left(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G$, denoted for short by $V_\chi$, one for each irreducible character $\chi \in \operatorname{Irr}(G)$. We remind to Section 4 where it makes more clear the meaning of these subspaces. We need the following

**Property** (#)**:** A product-quotient surface $S$ satisfies Property (#) if

$$\dim V_\chi \neq 0 \implies \deg(\chi) = 1$$

for each $\chi \in \operatorname{Irr}(G)$.

**Remark 0.3.** Property (#) always holds for $G$ abelian group, since each irreducible character of $G$ has degree 1.

Assume that $S$ satisfies Property (#). Then Corollary 4.23 gives a formula of the base locus of each linear subsystem associated to the subspace $V_\chi$, $\chi \in \mathrm{Irr}(G)$, and so of the base locus of $H^{2,0}(C_1 \times C_2)^G$ by intersecting them. In other words, Property (#) makes computable the degree of the canonical map of $S$ automatically, see Subsection 4.6 for an example.

Furthermore, Corollary 4.23 also implies that Property (#) makes the canonical system $|K_S|$ of $S$ spanned by $p_g$ divisors that are union of fibres (with multiplicity) for the natural fibrations $S \to C_i$, $i = 1, 2$.

Thus, we have used the results obtained in Section 4 to produce a MAGMA code that automatically computes the degree of the canonical map of a product-quotient surface $S$ with $q = 0$ and $p_g = 3$ satisfying Property (#).

We have then selected those surfaces in Theorem 0.2 satisfying Property (#) and we have computed the degree of their canonical map. We have obtained a series of examples that are listed in Table 1. The numbers of column *no.* of Table 1 are referred to the row number of tables 17 to 29 in the appendix. We refer to the appendix of this paper where we explain in details all the other information contained in the columns of Table 1.

The paper is organized as follows:
In Section 1 we discuss finite group actions on a product of Riemann surfaces. We then present the main Theorem 1.19, the extended version of Theorem 0.1 of the introduction, crucial to speed up the classification algorithm.

Roughly speaking, Theorem 1.19 plays a crucial role in determining the irreducible families of surfaces of Theorem 0.2. Techniques to establish whether a pair of product-quotient surfaces belong to the same irreducible family have been extensively studied first in [1, Thm. 1.3] and [2, Prop. 5.2] in the case of surfaces isogenous to a product, and then in the general case in [8].
Theorem 1.19 seems to be a relevant new result on this problem, very useful in overcoming the huge amount of calculations that usually occur when adopting those techniques. We refer also to the last Section 5 where, among other things, we show the efficiency of our MAGMA code by using Theorem 1.19.

Apart from the rows where $N$ of families is denoted by ?, and whose challenges are discussed in Remark 5.6, the classification outlined in Theorem 0.2 yields a total of 1502 irreducible families of minimal surfaces of general type. Additionally, each family with $K^2 = 32$ maps onto an irreducible component (in the Zariski topology) of the Gieseker moduli space $\mathfrak{M}_{(4,32)}$, which consists of surfaces of general type with $K_S^2 = 32$ and $\chi(\mathcal{O}_S) = 4$. The remaining cases, where $23 \leq K^2 \leq 30$, are more delicate and we refer to Subsection 1.2 and Remark 1.14.

In Section 2 we generalize [8, Prop. 1.14] to any $\chi \in \mathbb{N}$ and discuss the classification algorithm.

In Section 3 we prove Theorem 0.2. In particular, we show that all surfaces of Theorem 0.2 are of general type and those in tables 17 to 29 are also minimal. We also discuss the exceptional cases arising from the secondary output of the function

| *no.* | $K_S^2$ | Sing$(X)$ | $t_1$ | $t_2$ | $G$ | *Id* | $N$ | $\deg(\Phi_S)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 32 | | $2^6$ | $2^8$ | $\mathbb{Z}_2^3$ | $\langle 8,5\rangle$ | 3 | $8, 16^2$ |
| 2 | 32 | | $2^5$ | $2^{12}$ | $\mathbb{Z}_2^3$ | $\langle 8,5\rangle$ | 3 | 0,4,8 |
| 3 | 32 | | $3^4$ | $3^7$ | $\mathbb{Z}_3^2$ | $\langle 9,2\rangle$ | 2 | 6, 12 |
| 4 | 32 | | $3^5$ | $3^5$ | $\mathbb{Z}_3^2$ | $\langle 9,2\rangle$ | 1 | 9 |
| 5 | 32 | | $2^3, 4^2$ | $2^3, 4^2$ | $G(16,3)$ | $\langle 16,3\rangle$ | 2 | 16 |
| 7 | 32 | | $2^2, 4^2$ | $2^5, 4^2$ | $G(16,3)$ | $\langle 16,3\rangle$ | 6 | 8 |
| 9 | 32 | | $2^3, 4$ | $2^{12}$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 6 | 0 |
| 12 | 32 | | $2^6$ | $2^6$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 1 | 32 |
| 13 | 32 | | $2^5$ | $2^8$ | $\mathbb{Z}_2^4$ | $\langle 16,14\rangle$ | 13 | $0, 8^5, 16^7$ |
| 14 | 32 | | $2^6$ | $2^6$ | $\mathbb{Z}_2^4$ | $\langle 16,14\rangle$ | 6 | $8, 16^3, 32^2$ |
| 21 | 32 | | $2^2, 4^2$ | $2^3, 4^2$ | $G(32,22)$ | $\langle 32,22\rangle$ | 7 | 16 |
| 28 | 32 | | $2^5$ | $2^6$ | $\mathbb{Z}_2^2 \times D_4$ | $\langle 32,46\rangle$ | 4 | 24 |
| 42 | 32 | | $7^3$ | $7^3$ | $\mathbb{Z}_7^2$ | $\langle 49,2\rangle$ | 7 | $0, 5, 7, 10, 11, 14^2$ |
| 48 | 32 | | $2^2, 4^2$ | $2^2, 4^2$ | $\mathbb{Z}_2^5 \rtimes \mathbb{Z}_2$ | $\langle 64,60\rangle$ | 3 | 32 |
| 87 | 30 | $1/2^2$ | $2^3, 4$ | $2^{10}, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 6 | 0 |
| 88 | 30 | $1/2^2$ | $2^4, 4$ | $2^5, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 2 | 4 |
| 119 | 28 | $1/2^4$ | $2^2, 4^2$ | $2^8, 4^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\langle 8,2\rangle$ | 1 | 0 |
| 120 | 28 | $1/2^4$ | $2^5$ | $2^{11}$ | $\mathbb{Z}_2^3$ | $\langle 8,5\rangle$ | 6 | $0^2, 4^3, 8$ |
| 123 | 28 | $1/2^4$ | $2^3, 4$ | $2^{11}$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 14 | 0 |
| 124 | 28 | $1/2^4$ | $2^5$ | $2^6, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 6 | 8 |
| 125 | 28 | $1/2^4$ | $2^2, 3^2$ | $3^4, 6^2$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18,3\rangle$ | 6 | $6^2$ |
| 198 | 26 | $1/2^6$ | $2^3, 4$ | $2^9, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 14 | 0 |
| 225 | 26 | $1/3^2, 2/3^2$ | $3, 9^2$ | $3^2, 9^2$ | $\mathbb{Z}_3 \times \mathbb{Z}_9$ | $\langle 27,2\rangle$ | 6 | $6^3, 7, 9, 10$ |
| 237 | 26 | $1/3^2, 2/3^2$ | $2, 6^2$ | $2^4, 6^2$ | $\mathbb{Z}_2^2 \times \mathcal{A}_4$ | $\langle 48,49\rangle$ | 5 | 8 |
| 283 | 24 | $1/2^8$ | $2^6$ | $2^{10}$ | $\mathbb{Z}_2^2$ | $\langle 4,2\rangle$ | 1 | 0 |
| 284 | 24 | $1/2^8$ | $2^3, 4^2$ | $2^4, 4^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\langle 8,2\rangle$ | 1 | 8 |
| 285 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^7, 4^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\langle 8,2\rangle$ | 1 | 2 |
| 286 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4^4$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\langle 8,2\rangle$ | 2 | 2, 8 |
| 289 | 24 | $1/2^8$ | $2^6$ | $2^7$ | $\mathbb{Z}_2^3$ | $\langle 8,5\rangle$ | 11 | $4^3, 6^2, 8^3, 12^2, 16$ |
| 290 | 24 | $1/2^8$ | $2^5$ | $2^{10}$ | $\mathbb{Z}_2^3$ | $\langle 8,5\rangle$ | 14 | $0^4, 4^7, 6, 8^2$ |
| 295 | 24 | $1/2^8$ | $2, 4^3$ | $4^4$ | $\mathbb{Z}_4^2$ | $\langle 16,2\rangle$ | 1 | 12 |
| 296 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4^2$ | $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$ | $\langle 16,3\rangle$ | 13 | $8^3$ |
| 298 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4^2$ | $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ | $\langle 16,10\rangle$ | 10 | $8^4, 12^4, 16^2$ |
| 303 | 24 | $1/2^8$ | $2^3, 4$ | $2^{10}$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 27 | 0 |
| 304 | 24 | $1/2^8$ | $2^5$ | $2^7$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 4 | 16 |
| 305 | 24 | $1/2^8$ | $2^4, 4$ | $2^6$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 14 | $8^2$ |
| 308 | 24 | $1/2^8$ | $2^5$ | $2^7$ | $\mathbb{Z}_2^4$ | $\langle 16,14\rangle$ | 13 | $8^5, 12^4, 16^4$ |
| 309 | 24 | $1/2^8$ | $2^2, 3^2$ | $3, 6^4$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18,3\rangle$ | 3 | 0, 6 |
| 312 | 24 | $1/2^8$ | $2, 3^4, 6$ | $2^2, 3^2$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18,3\rangle$ | 3 | 6 |
| 376 | 24 | $1/2^8$ | $2, 3^2, 6$ | $3, 6^2$ | $S_3 \times \mathbb{Z}_3^2$ | $\langle 54,12\rangle$ | 9 | 12, (16, 18), (13, 15), 18, 24 |
| 459 | 24 | $1/4^2, 3/4^2$ | $2^3, 4$ | $2^9, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 6 | 0 |
| 475 | 23 | $1/3^3, 2/3^3$ | $3^4$ | $3^6$ | $\mathbb{Z}_3^2$ | $\langle 9,2\rangle$ | 6 | $6^5, 9$ |
| 477 | 23 | $1/3^3, 2/3^3$ | $2^2, 3^2$ | $2^4, 3, 6$ | $\mathbb{Z}_2 \times \mathcal{A}_4$ | $\langle 24,13\rangle$ | 2 | 8 |
| 486 | 23 | $1/3^3, 2/3^3$ | $2, 6^2$ | $2^4, 3, 6$ | $\mathbb{Z}_2^2 \times \mathcal{A}_4$ | $\langle 48,49\rangle$ | 6 | 8 |

TABLE 1

*ListGroups*$(K^2, 4)$ for each $K^2 \in \{23, \dots, 32\}$ in order to obtain the complete list of tables 17 to 29 of the appendix.

In Section 4 we investigate the canonical map of product-quotient surfaces $S$ and we present an algorithm to compute its degree whenever $S$ satisfies Property (#). The examples of 1 with a high degree of the canonical map that are to our knowledge already discovered in the literature are the following:

- Surfaces of *no.*42 in Table 1 are the examples presented in [19];
- Families of surfaces *no.*376 having a degree of the canonical map 12, $(16, 18)$, $(13, 15)$, 18 are all those in [18]. Furthermore, we point out that only surfaces *no. 1* of [18, Thm. 2.3] satisfy Property (#) thanks to which the degree of their canonical map was automatically computable.
- One of the families of *no.*28 in Table 1 having 24 as degree of the canonical map is already studied by the author of the present paper and D. Frapporti and can be found in [17, Sec. 6.3]. We also mention that this family of surfaces does not satisfy Property (#), hence in this case we have had to find the equations of the pair of curves realizing that family of surfaces and then studied by hands the degree of their canonical map.
- Two of the six families of *no.*14 in Table 1 having degree 32 of the canonical map are discussed in [25]. They are described there differently from us, as $\mathbb{Z}_2^4$-coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ using the language of Pardini's theory of abelian coverings. Surfaces of these families are the only examples in the literature with a canonical map of degree 32, which is also the highest possible degree for product-quotient surfaces as observed in Remark 4.1.

  Furthermore, the authors proved in [25, Prop. 5.3] that these two examples are the only product-quotient surfaces with $G$ abelian having degree of the canonical map equal to 32. The same question with $G$ not abelian was still-open and it find an answer in the present paper. Indeed, there are other families of surfaces in Table 1 with a canonical map of degree 32.

Section 5 is devoted to comments about the computational complexity of the presented algorithms.

An expanded version of Tables 17 to 29 of the appendix describing all the needed data to work explicitly with one of the surfaces and a commented version of the MAGMA codes we used can be found here:
`https://github.com/Fefe9696/PQ_Surfaces_with_fixed_Ksquare_chi`

## Notation

We will use the basic notations of the theory of smooth complex projective surfaces, hence $K_S$ is the *canonical class* of $S$, $p_g := h^0(S, K_S)$ is the *geometric genus*, $q(S) := h^1(S, \mathcal{O}_S)$ is the *irregularity*, and $\chi(\mathcal{O}_S) = 1 - q + p_g$ is the *Euler characteristic*.

# 1. Algebraic characterization of families of product-quotient surfaces given by a pair of $G$-coverings of $\mathbb{P}^1$

Let us consider a finite group $G$ acting on two smooth projective curves $C_1$ and $C_2$ of respective genera at least 2. We consider the diagonal action of $G$ on $C_1 \times C_2$. In this case the action of $G$ on $C_1 \times C_2$ is called *unmixed*. Following [14, Remark 3.10], we can assume that the action on $C_i$ is faithful.

**Definition 1.1.** [8, Defn. 0.1] The minimal resolution of singularities $S$ of $X = (C_1 \times C_2)/G$ is called *product-quotient* surface of the *quotient model* $X$.

Let $S$ be a product-quotient surface of quotient model $(C_1 \times C_2)/G$. From a theorem due to Serrano [33, Prop. 2.2], then $q(S) = 0$ if and only if $C_i/G \cong \mathbb{P}^1$. In other words, pairs of $G$-coverings of the projective line defines regular product-quotient surfaces. For this reason, let us briefly recall how coverings of $\mathbb{P}^1$ can be described.

## 1.1. Algebraic characterization of families of $G$-coverings of $\mathbb{P}^1$.

**Definition 1.2.** Let $G$ be a finite group. For a $G$-covering of $\mathbb{P}^1$ we mean a Riemann surface $C$ together with a (holomorphic) action $\phi$ of $G$ on $C$ such that the quotient $C/G$ is $\mathbb{P}^1$. Whenever we need to specify the action, we write $(C, \phi)$.

There are two notions of equivalence among $G$-coverings of $\mathbb{P}^1$: we say that $C_1$ and $C_2$ are *topologically equivalent* if there exists a orientation preserving homeomorphism $f \colon C_1 \to C_2$ and an automorphism $\varphi \in \mathrm{Aut}(G)$ such that $f(g \cdot p) = \varphi(g) \cdot f(p)$ for any $g \in G$ and $p \in C_1$. We say that $C_1$ and $C_2$ are *isomorphic* if moreover $f$ is a biholomorphism.

Consider the set of G-coverings of $\mathbb{P}^1$ modulo isomorphism. The topological equivalence partitions it into equivalence classes, let $\mathcal{C}$ be one of them. González-Díez and Harvey showed in [26] that $\mathcal{C}$ has a natural structure of connected complex manifold such that the natural map of $\mathcal{C}$ on the moduli space of curves mapping $(C, \phi)$ to $C$ is analytic. More precisely, the manifold $\mathcal{C}$ is the normalization of its image $\widetilde{\mathcal{C}}$. In particular, $\widetilde{\mathcal{C}}$ is always an irreducible subvariety of the moduli space of curves.

The manifold $\mathcal{C}$ can be realized by taking a $G$-covering $C \in \mathcal{C}$ and moving the branch points of its covering map $C \to \mathbb{P}^1$. Each move of the branch points of $C$ corresponds to a topological covering of a complement of $\mathbb{P}^1$ given by those points. Thus, from the Riemann Existence Theorem, $C$ can be endowed with a new structure as Riemann surface. Hence we obtain another $G$-covering of $\mathbb{P}^1$ always topological equivalent but possibly not isomorphic to $C$. More precisely, they are not isomorphic if the move of the branch points is not a projective transformation of $\mathbb{P}^1$. In other words, we are realizing $G$-coverings of $\mathcal{C}$ not isomorphic to $C$ whenever we fix three of the branch points of $C$ and move the remaining ones.

From this we may easily deduce the dimension of the complex manifold $\mathcal{C}$, which is $r-3$, where $r$ is the number of branch points of a $G$-covering $C \in \mathcal{C}$.

**Definition 1.3.** We set $\mathcal{T}^r(G)$ be the collection of all classes of $G$-coverings of $\mathbb{P}^1$ ramified over $r$ points modulo topological equivalence.

From the above discussion, we invite the reader to think of each element of $\mathcal{T}^r(G)$ as a class $\mathcal{C}$ of families of $G$-coverings of $\mathbb{P}^1$ two-by-two not isomorphic but all topological equivalent to each other.

We shall give an algebraic description of the elements of $\mathcal{T}^r(G)$.

**Definition 1.4.** A *spherical system of generators* (of length $r$) of $G$ is a sequence $[g_1, \dots, g_r] \in G^r$ of elements of $G$ such that $g_i \neq 1$ for all $i$, and

- $G = \langle g_1, \dots, g_r \rangle$;
- $g_1 \cdots g_r = 1$.

The sequence $[o(g_1), \dots, o(g_r)]$ is called *signature* of $[g_1, \dots, g_r]$.

**Definition 1.5.** We set $\mathcal{D}^r(G) \subset G^r$ be the collection of all spherical systems of generators of $G$ of length $r$.

**Remark 1.6.** For each signature $[m_1, \dots, m_r]$ consider the *orbifold group*

$$\mathbb{T}(m_1, \dots, m_r) := \langle \gamma_1, \dots, \gamma_r | \gamma_1^{m_1}, \dots, \gamma_r^{m_r}, \gamma_1 \cdots \gamma_r \rangle.$$

There is a natural bijection among the set of surjective homomorphisms $\mathbb{T}(m_1, \dots, m_r) \to G$ and the set of spherical systems of generators of signature $[m_1, \dots, m_r]$.

The bijection associates to any homomorphism $\varphi$ the spherical system of generators $[\varphi(\gamma_1), \dots, \varphi(\gamma_r)]$.

Consider the braid group $\mathcal{B}_r$, whose presentation with generators $\sigma_1, \dots, \sigma_{r-1}$ is

$$\mathcal{B}_r = \left\langle \sigma_1, \dots, \sigma_{r-1} \colon \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i, & |i-j| > 1 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, & |i-j| = 1 \end{array} \right\rangle.$$

The elements of $\mathcal{B}_r$ are called *Hurwitz moves*. We consider the following action of $\operatorname{Aut}(G) \times \mathcal{B}_r$ on $\mathcal{D}^r(G)$:

$$\Psi \cdot [g_1, \dots, g_r] := [\Psi(g_1), \dots, \Psi(g_r)], \qquad \Psi \in \operatorname{Aut}(G).$$

$$\sigma_i \cdot [g_1, \dots, g_r] := [g_1, \dots, g_{i-1}, g_i \cdot g_{i+1} \cdot g_i^{-1}, g_i, g_{i+2}, \dots g_r], \qquad \sigma_i \in \mathcal{B}_r.$$

The action of the generators $\sigma_i$ extends to an action of the entire $\mathcal{B}_r$.
We finally have the following classical result

**Theorem 1.7.** *The collection of all classes of $G$-coverings of $\mathbb{P}^1$ ramified over $r$ points modulo topological equivalence is in bijection with $\mathcal{D}^r(G)/\operatorname{Aut}(G) \times \mathcal{B}_r$:*

$$\mathcal{T}^r(G) \cong \mathcal{D}^r(G)/\operatorname{Aut}(G) \times \mathcal{B}_r. \tag{1.1}$$

For a recent proof, we refer to [22, Cor. 5.7].

We briefly describe the bijection in Theorem 1.7. Take an element in the quotient $\mathcal{D}^r(G)/\operatorname{Aut}(G) \times \mathcal{B}_r$, and choose a representative $[g_1, \dots, g_r]$ of it. From Remark 1.6 we obtain a surjective morphism $\mathbb{T}(m_1, \dots, m_r) \to G$, with $m_i = o(g_i)$. We choose a finite set $X := \{q_1, \dots, q_r\}$ on $\mathbb{P}^1$, a point $q_0 \in \mathbb{P}^1 \setminus X$, and a *geometric basis* of the fundamental group of $\mathbb{P}^1 \setminus X$ with base point $q_0$, which is a set of $r$ distinct homotopy class loops $\eta_i$ of $\mathbb{P}^1 \setminus X$ starting at $q_0$ and travelling around $q_i$ in the counterclockwise order, $i = 1, \dots, r$.

Observe that any class loop of $\pi_1(\mathbb{P}^1 \setminus X, q_0)$ is generated by $\eta_1, \ldots, \eta_r$ and the only relation among $\eta_1, \ldots, \eta_r$ is that their product $\eta_1 \cdots \eta_r$ can be contracted to $q_0$. In other words, we have obtained $\pi_1(\mathbb{P}^1 \setminus X, q_0)$ has presentation $\langle \gamma_1, \ldots, \gamma_r | \gamma_1 \cdots \gamma_r \rangle$.

We observe then $\mathbb{T}(m_1, \ldots, m_r)$ is a quotient of $\pi_1(\mathbb{P}^1 \setminus X, q_0)$, and the kernel of the composition

$$\pi_1(\mathbb{P}^1 \setminus X, q_0) \to \mathbb{T}(m_1, \ldots, m_r) \to G$$

defines a unique topological $G$-covering of $\mathbb{P}^1 \setminus X$. By Riemann Existence theorem, this completes to a $G$-covering $C$ of $\mathbb{P}^1$.

The bijection of Theorem 1.7 maps the class of $[g_1, \ldots, g_r]$ modulo $\mathrm{Aut}(G) \times \mathcal{B}_r$ to the class of $C$ modulo topological equivalence.

In particular, Theorem 1.7 says that

(1) if in the above construction we change
   - the set of spherical generators $[g_1, \ldots, g_r]$ by a set in the same orbit for the action of $\mathrm{Aut}(G) \times \mathcal{B}_r$, or
   - the points $q_0, q_1, \ldots, q_r$ with other $r+1$ points of $\mathbb{P}^1$, or
   - the choice of the geometric basis $\eta_1, \ldots, \eta_r$

   then the new obtained $G$-covering is topologically equivalent to $C$;
(2) if $C_1$ and $C_2$ are obtained by spherical systems of generators that are not in the same $\mathrm{Aut}(G) \times \mathcal{B}_r$ orbit then $C_1$ and $C_2$ are not topologically equivalent.
(3) every $G$-covering $C$ of $\mathbb{P}^1$ ramified over $r$ points up to topological equivalence is obtained in this way by a spherical system of generators of $G$.

We need to discuss only the third point above. Hence we show how to get a spherical system of generators from a $G$-covering of the projective line.
Given a $G$-covering $\lambda\colon C \to \mathbb{P}^1$ whose branch locus consists of $r$ points $X := \{q_1, \ldots, q_r\}$, then $\lambda\colon C \setminus \lambda^{-1}(X) \to \mathbb{P}^1 \setminus X$ is a topological covering. Chosen a point $p_0$ over $q_0 \in \mathbb{P}^1 \setminus X$, we consider the monodromy map $\pi_1(\mathbb{P}^1 \setminus X) \to \lambda^{-1}(p_0)$. Since $q_0$ is not a branch point of the covering, then $\lambda^{-1}(q_0)$ consists of $|G|$ points so that we can identity $\lambda^{-1}(q_0)$ with $G$: $g \cdot p_0 \mapsto g$. In this way, the monodromy map becomes a morphism of groups.
Notice that only the kernel of this map is uniquely determined by the covering.
Each class loop of the geometric basis $\eta_1, \ldots, \eta_r$ is sent to elements $g_1, \ldots, g_r$ via the monodromy map. Since the monodromy map is surjective, then $g_1, \ldots, g_r$ generate $G$, whilst the contraction of $\eta_1 \cdots \eta_r$ to the point $q_0$ translates as $g_1 \cdots g_r = 1_G$. Thus, $[g_1, \ldots, g_r]$ is a spherical system of generators of $G$.

This construction makes also very clear the geometrical meaning of the elements $g_1, \ldots, g_r$. Indeed, we see that $g_i$ is a generator of the stabilizer subgroup of a point $p_i$ over $q_i$ and there is a local coordinate $z$ on $C$ around $p_i$ such that the action of $g_i$ sends a point of coordinate $z$ to the point of coordinate $e^{\frac{2\pi i}{m_i}} z$, where $m_i$ is the ramification index of $p_i$.
In other words, $g_i$ is the *local monodromy* of a point over $q_i$, see [17, Sec. 2.1.1] for more details.

**Remark 1.8.** Notice that from the above discussion we have also obtained the order of $g_i$ is simply the ramification index $m_i$ of $q_i$.
Hence the bijection of Theorem 1.7 sends the class $[g_1, \dots, g_r]$ modulo $\mathrm{Aut}(G)\times\mathcal{B}_r$ to the class of a $G$-covering $C$ modulo topological equivalence branched on $r$ points $\{q_1, \dots, q_r\}$ with ramification indices $o(g_1), \dots, o(g_r)$ respectively and such that the Hurwitz formula holds:

$$2g(C) - 2 = |G| \left( -2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{o(g_i)} \right) \right). \tag{1.2}$$

Here the cyclic groups $\langle g_i \rangle$ (and their conjugates) provide the non-trivial stabilizers of the action of $G$ on $C$.

Let us give a example of how to use Theorem 1.7.

**Example 1.9.** We are going to compute $\mathcal{T}^3(S_3\times\mathbb{Z}_3^2)$, the collection of the $S_3\times\mathbb{Z}_3^2$-coverings of $\mathbb{P}^1$ up to topological equivalence ramified over 3 points. Up to apply suitable Hurwitz moves, we can assume that a spherical system of generators $[(g_1, v_1), (g_2, v_2), (g_3, v_3)]$ has $o(g_1) \le o(g_2) \le o(g_3)$. Observe $g_i \neq 1$, otherwise $S_3$ would be generated by only one element, and this is not possible since it is not cyclic. The same argument holds for $\mathbb{Z}_3^2$, so that $v_i \neq 0$. This implies $[g_1, g_2, g_3] \in \mathcal{D}^3(S_3)$, and $[v_1, v_2, v_3] \in \mathcal{D}^3(\mathbb{Z}_3^2)$. By Hurwitz formula 1.2, then $\sum_{i=1}^{3} \frac{1}{o(g_i)}$ has to be an integer, which holds only either for $o(g_1) = o(g_2) = o(g_3) = 3$ or $o(g_1) = o(g_2) = 2$, and $o(g_3) = 3$. The first case can be excluded since there are no $g_1, g_2, g_3$ of order 3 generating $S_3$.
Let us focus on the second case, which gives $g(C) = 0$, so $C \cong \mathbb{P}^1$. The elements of order 2 of $S_3$ are $\tau, \tau\sigma$, and $\tau\sigma^2$, where $\tau$ is a reflection (transposition) and $\sigma$ is a rotation (3-cycle) of $S_3$. Since $g_3 = g_2^{-1}g_1^{-1}$, then $g_1 \neq g_2$ otherwise $g_3 = 1$ since $g_1$ and $g_2$ has order two.
Thus the list of spherical systems with ordered signature $[2, 2, 3]$ consists only of six elements obtained just by choosing a distinct pair of $g_1, g_2$ in the set $\{\tau, \tau\sigma, \tau\sigma^2\}$. From here it is easy to see that the action of $\mathrm{Aut}(S_3)$ on $\mathcal{D}^3(S_3)$ is transitive.

From the other side, it is clear that the action of $\mathrm{GL}_2(\mathbb{Z}_3)$ on $\mathcal{D}^3(\mathbb{Z}_3^2)$ is transitive. Thus $\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)$ acts transitively on $\mathcal{D}^3(S_3 \times \mathbb{Z}_3^2)$ and from Theorem 1.7 we obtain

$$\mathcal{T}^3(S_3 \times \mathbb{Z}_p^2) \cong \frac{\mathcal{D}^3(S_3 \times \mathbb{Z}_3^2)}{\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2) \times \mathcal{B}_3} = \{[(\tau, (1, 0)), (\tau\sigma, (0, 1)), (\sigma^2, (p-1, p-1))]\}.$$

By the Hurwitz formula (1.2), the genus of the corresponding $G$-covering $C$ is $g(C) = 10$.
Here $C$ may be described explicitly by equations as follows: we consider the projective space $\mathbb{P}^3$ with homogeneous coordinates $x_0, \dots, x_3$ and define

$$C \colon \begin{cases} x_2^3 = x_0^3 - x_1^3 \\ x_3^3 = x_0^3 + x_1^3 \end{cases}.$$

The action $\phi\colon S_3 \times \mathbb{Z}_3^2 \to \mathrm{Aut}(C)$ is given by

$$\big(\sigma^i\tau^j, (a, b)\big) \mapsto [(x_0 : x_1 : x_2 : x_3) \mapsto (\zeta_3^i x_{[j]} : x_{[j+1]} : (-1)^j \zeta_3^{2a+2i} x_2 : \zeta_3^{2b+2i} x_3)],$$

where $\zeta_3 := e^{\frac{2\pi i}{3}}$ is the first 3-root of the unity. Finally, the covering map by this action is

$$\lambda\colon C \xrightarrow{9:1} \mathbb{P}^1 \xrightarrow{6:1} \mathbb{P}^1, \qquad (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1) \mapsto \left(x_0^3 x_1^3 : (x_0^6 + x_1^6)/2\right).$$

**Remark 1.10.** As we could expect, it becomes soon computationally difficult getting the $\mathrm{Aut}(G) \times \mathcal{B}_r$-orbits of $\mathcal{D}^r(G)$, when $r$ or $G$ increase. For this reason, several authors put an increased effort into the development of an efficient algorithm to compute such orbits, usually with the helping also of a computational algebra system (e.g. MAGMA, [13]). A big step forward in this direction is given for instance in [16], where the authors collect in a database a representative for any orbit of spherical systems of generators of fixed genus $g \leq 40$, with a few exceptions.
We use this database and their script *FindGenerators* to speed up **Step 3** of Subsection 2.1 and in combination with Theorem 1.19 to give improvements in **Step 5**.

1.2. **Families of product-quotient surfaces from a pair of coverings of $\mathbb{P}^1$.** In this subsection we study how to realize all families of product-quotient surfaces obtained by a pair of topological types of $G$-coverings of $\mathbb{P}^1$.

**Definition 1.11.** Let us call by $\mathcal{T}^{r,s}(G)$ the collection of all families of regular product-quotient surfaces, whose natural fibrations $\lambda_i$ are $G$-coverings $C_i$ of $\mathbb{P}^1$ branched over $r$ and $s$ points respectively.

**Remark 1.12.** In the above definition the order of $C_1$ and $C_2$ is relevant. Thus exchanging them gives a natural bijection $\iota\colon \mathcal{T}^{r,s}(G) \to \mathcal{T}^{s,r}(G)$ which sends families to isomorphic families of surfaces.

We give a generalization of Theorem 1.7 for product-quotient surfaces (see [8] and [3] for more details).

**Proposition 1.13.** *There is a natural bijection among $\mathcal{T}^{r,s}(G)$ and*

$$\frac{\mathcal{D}^r(G) \times \mathcal{D}^s(G)}{\mathrm{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s},$$

*where $\mathrm{Aut}(G)$ acts simultaneously on both factors, whilst $\mathcal{B}_r$ and $\mathcal{B}_s$ act on the first and second factor respectively.*

**Remark 1.14.** We point out that each of the families of $\mathcal{T}^{r,s}(G)$ maps onto an algebraic subset of the Gieseker moduli space, but the images of two different families may not be distinct. This because we are considering an equivalence relation among product-quotient surfaces that is weaker than the equivalence relation *being isomorphic*.
However, as proved in [2, Prop. 5.2], regular product-quotient surfaces $S_1$ and $S_2$ isogenous to a product are isomorphic if either their algebraic data share the same orbit by the action of $\mathrm{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s$ or the isomorphism is given by exchanging the factors $C_1$ and $C_2$.

The bijection in Proposition 1.13 is given by a map $\mathcal{D}^r(G) \times \mathcal{D}^s(G) \to T^{r,s}(G)$ as follows.

Consider a pair of spherical systems of generators $[g_1, \dots, g_r]$ and $[h_1, \dots, h_s]$. We fix points $q_0, q_1, \dots, q_r \in \mathbb{P}^1$ and a geometric basis $\eta_1, \dots, \eta_r$ as in the Subsection 1.1, where $\eta_i$ is a class loop based at $q_0$ travelling around $q_i$ in the counter-clockwise order. In this way, following the description of Theorem 1.7 in Subsection 1.1 we get from the first spherical system $[g_1, \dots, g_r]$ a $G$-covering of the line $\lambda_1 \colon C_1 \to \mathbb{P}^1$ whose branch locus is $\{q_1, \dots, q_r\}$, where $q_i$ has ramification index $o(g_i)$ and $g_i$ is the local monodromy of a point over $q_i$.

A similar argument holds for $[h_1, \dots, h_s]$, so we get another $G$-covering $\lambda_2 \colon C_2 \to \mathbb{P}^1$ branched over $s$ points with ramification indices $o(h_1), \dots, o(h_s)$ and $h_i$ is the local monodromy of a point over $q_i$.

The diagonal action of $G$ on $C_1 \times C_2$ gives a product-quotient surface $S$ whose quotient model is $(C_1 \times C_2)/G$.

The map $\mathcal{D}^r(G) \times \mathcal{D}^s(G) \to T^{r,s}(G)$ sends the pair of spherical systems $([g_1, \dots, g_r], [h_1, \dots, h_s])$ to the family of $S$.

Let us discuss how $\operatorname{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s$ acts on this construction.

We show that acting with $\Psi \in \operatorname{Aut}(G)$ simultaneously on $[g_1, \dots, g_r]$ and $[h_1, \dots, h_s]$ the isomorphic class of $S$ does not change. Acting with $\Psi$, we obtain the same $G$-coverings $C_1$ and $C_2$, but the isomorphisms among $G$ and the automorphism groups of $C_1$ and $C_2$ are both modified by composition with $\Psi$. Then we obtain the same product $C_1 \times C_2$ and the action of $G \times G$ on it has been modified by composition with $\Psi \times \Psi$. Since $\Psi \times \Psi$ sends the diagonal to itself, then we obtain a surface isomorphic to $S$.

The group $\mathcal{B}_r$ acts only on the first spherical system of generators $[g_1, \dots, g_r]$ replacing $C_1$ with a topological equivalent $G$-covering $C_1'$ as described in Subsection 1.1. By the result of González-Díez and Harvey in [26] mentioned there, then $C_1$ and $C_1'$ are in the same irreducible connected family of $G$-coverings. In particular, the action of $\mathcal{B}_r$ on the given construction connects surfaces of the same family.

An analogous statement holds for the action of $\mathcal{B}_s$ on a spherical system of generators $[h_1, \dots, h_s]$.

To each family of product-quotient surfaces we have a naturally associated pair of topological types of $G$-coverings, thus giving a surjective map $\mathcal{T}^{r,s}(G) \twoheadrightarrow \mathcal{T}^r(G) \times \mathcal{T}^s(G)$. By Proposition 1.13 and Theorem 1.7 we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}^{r,s}(G) & \longleftrightarrow & \frac{\mathcal{D}^r(G)\times\mathcal{D}^s(G)}{\operatorname{Aut}(G)\times\mathcal{B}_r\times\mathcal{B}_s} \\ \big\downarrow & & \big\downarrow \pi \\ \mathcal{T}^r(G)\times\mathcal{T}^s(G) & \longleftrightarrow & \frac{\mathcal{D}^r(G)}{\operatorname{Aut}(G)\times\mathcal{B}_r} \times \frac{\mathcal{D}^s(G)}{\operatorname{Aut}(G)\times\mathcal{B}_s} \end{array} \tag{1.3}$$

Here $\pi$ is defined as the only map making the diagram commutative. Such $\pi$ sends the class of a pair of spherical systems of generators $[V_1, V_2]$ to the pair of classes $([V_1], [V_2])$.

We are going to find the inverse image of each point $([V_1], [V_2])$ by $\pi$, which translates in determining each family of product-quotient surfaces afforded by the pair of topological types of $G$-coverings, the first given by $[V_1]$, and the second by $[V_2]$.

**Definition 1.15.** Let $V$ be a spherical system of generators of length $r$ of a finite group $G$. The group of automorphisms of *braid type* on $V$ is the following subgroup of $\operatorname{Aut}(G)$

$$\mathcal{B}\operatorname{Aut}(G, V) := \{\varphi \in \operatorname{Aut}(G) \colon \exists\, \sigma \in \mathcal{B}_r \text{ such that } \varphi \cdot V = \sigma \cdot V\}.$$

Since the action of an automorphism of $G$ commutes with the action of a braid on a spherical system of generators, then is immediate to see $\mathcal{B}\operatorname{Aut}(G, V)$ is subgroup of $\operatorname{Aut}(G)$: given $\varphi_1, \varphi_2 \in \mathcal{B}\operatorname{Aut}(G, V)$, then

$$\left(\varphi_1 \circ \varphi_2^{-1}\right) \cdot V = \varphi_1(\sigma_2^{-1} \cdot V) = \sigma_2^{-1} \cdot (\varphi_1 \cdot V) = (\sigma_2^{-1}\sigma_1) \cdot V$$

for some $\sigma_1, \sigma_2 \in \mathcal{B}_r$. Thus $\varphi_1 \circ \varphi_2^{-1} \in \mathcal{B}\operatorname{Aut}(G, V)$.

**Remark 1.16.** If you replace $V$ by $V'$ on its $\operatorname{Aut}(G) \times \mathcal{B}_r$-orbit, let us say $V' := (\Psi, \sigma) \cdot V$, then the subgroup $\mathcal{B}\operatorname{Aut}(G, V')$ is conjugate to $\mathcal{B}\operatorname{Aut}(G, V)$:

$$\mathcal{B}\operatorname{Aut}(G, V') = \Psi \circ \mathcal{B}\operatorname{Aut}(G, V) \circ \Psi^{-1}.$$

Note that $\Psi \in \mathcal{B}\operatorname{Aut}(G, V)$ implies $\mathcal{B}\operatorname{Aut}(G, V') = \mathcal{B}\operatorname{Aut}(G, V)$.

**Definition 1.17.** Let $V_1$ and $V_2$ be a pair of spherical systems of generators of $G$. We will say that two automorphisms $\Phi, \Psi \in \operatorname{Aut}(G)$ are $(V_1, V_2)-$related, and we will write

$$\Phi \sim_{V_1, V_2} \Psi$$

if there exist $\varphi_1 \in \mathcal{B}\operatorname{Aut}(G, V_1), \varphi_2 \in \mathcal{B}\operatorname{Aut}(G, V_2)$ such that

$$\Psi = \varphi_1 \circ \Phi \circ \varphi_2.$$

The relation $\sim_{V_1, V_2}$ is clearly an equivalence relation on $\operatorname{Aut}(G)$. We denote by $Q\operatorname{Aut}(G)_{V_1, V_2}$ the quotient of $\operatorname{Aut}(G)$ by $\sim_{V_1, V_2}$.

In other words $Q\operatorname{Aut}(G)_{V_1, V_2}$ is the set of double cosets

$$Q\operatorname{Aut}(G)_{V_1, V_2} = \mathcal{B}\operatorname{Aut}(G, V_1) \backslash \operatorname{Aut}(G) / \mathcal{B}\operatorname{Aut}(G, V_2).$$

**Remark 1.18.** Replacing $V_1$ and $V_2$ by two spherical systems of generators in the same orbits, namely $V_1' = (\Psi_1, \sigma_1) \cdot V_1$ and $V_2' = (\Psi_2, \sigma_2) \cdot V_2$, then by Remark 1.16 we have

$$\Phi \sim_{V_1, V_2} \Psi \iff \Psi_1 \circ \Phi \circ \Psi_2^{-1} \sim_{V_1', V_2'} \Psi_1 \circ \Psi \circ \Psi_2^{-1}.$$

Moreover, the bijection $\Phi \mapsto \Psi_1 \circ \Phi \circ \Psi_2^{-1}$ induces a bijection among the quotients

$$Q\operatorname{Aut}(G)_{V_1, V_2} \leftrightarrow Q\operatorname{Aut}(G)_{V_1', V_2'}, \qquad [\Phi] \mapsto [\Psi_1 \circ \Phi \circ \Psi_2^{-1}] \tag{1.4}$$

that only depends on $V_1, V_2, V_1'.V_2'$ and not on the choice of $\Psi_1, \Psi_2$.

We can finally state and prove the main theorem of this section:

**Theorem 1.19.** *We consider the map* $\pi$ *defined at* (1.3)*. Let us fix a point* $x \in \frac{\mathcal{D}^r(G)}{Aut(G)\times\mathcal{B}_r} \times \frac{\mathcal{D}^s(G)}{Aut(G)\times\mathcal{B}_s}$*, and let us choose a pair of spherical systems of generators* $V_1$ *and* $V_2$ *such that* $x = ([V_1],[V_2])$*. The following hold:*

*(1) given* $\Phi \in \operatorname{Aut}(G)$*, then*

$$[V_1, \Phi\cdot V_2] \in \frac{\mathcal{D}^r(G)\times\mathcal{D}^s(G)}{\operatorname{Aut}(G)\times\mathcal{B}_r\times\mathcal{B}_s}$$

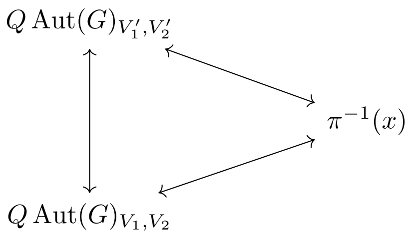
*depends only by class of* $\Phi$ *in* $Q\operatorname{Aut}(G)_{V_1,V_2}$*.*

*(2) The map*

$$\begin{aligned} Q\operatorname{Aut}(G)_{V_1,V_2} &\longrightarrow \pi^{-1}(x) \\ [\Phi] &\mapsto [V_1, \Phi\cdot V_2] \end{aligned} \tag{1.5}$$

*is bijective. In particular,* $|\pi^{-1}(x)| = |Q\operatorname{Aut}(G)_{V_1,V_2}|$*.*

*(3) If we replace* $V_1$ *by* $V_1'$ *in the same* $\operatorname{Aut}(G)\times\mathcal{B}_r$*-orbit, and* $V_2$ *by* $V_2'$ *in the same* $\operatorname{Aut}(G)\times\mathcal{B}_s$*-orbit, then the bijective maps in* (1.4) *and* (1.5) *form a commutative triangle*




*Proof.* (1) Let us consider an automorphism $\Phi' = \varphi_1\circ\Phi\circ\varphi_2$ in the same class of $\Phi$ in $Q\operatorname{Aut}(G)_{V_1,V_2}$, where $\varphi_1 \in \mathcal{B}\operatorname{Aut}(G,V_1)$ and $\varphi_2 \in \mathcal{B}\operatorname{Aut}(G,V_2)$. Then

$$\begin{aligned} [V_1, \Phi'\cdot V_2] &= [V_1, (\varphi_1\circ\Phi\circ\varphi_2)\, V_2] \\ &= [\varphi_1^{-1}\cdot V_1, (\Phi\circ\varphi_2)\cdot V_2] \\ &= [\sigma_1^{-1}\cdot V_1, \Phi\cdot(\sigma_2\cdot V_2)] \\ &= [\sigma_1^{-1}\cdot V_1, \sigma_2\cdot(\Phi\cdot V_2)] = [V_1, \Phi\cdot V_2]. \end{aligned}$$

(2) Point 1. proves that the map 1.5 is well-defined. Let us consider an element $[V_1', V_2'] \in \pi^{-1}(x)$, hence $V_1'$ is in the same orbit of $V_1$ and $V_2'$ is in the same orbit of $V_2$. We write

$$V_1' = (\Psi_1,\sigma_1)\cdot V_1 \qquad \text{and} \qquad V_2' = (\Psi_2,\sigma_2)\cdot V_2,$$

where $(\Psi_1,\sigma_1) \in \operatorname{Aut}(G)\times\mathcal{B}_r$, and $(\Psi_2,\sigma_2) \in \operatorname{Aut}(G)\times\mathcal{B}_s$. Then

$$[V_1', V_2'] = [\Psi_1\cdot V_1, \Psi_2\cdot V_2] = [V_1, \left(\Psi_1^{-1}\circ\Psi_2\right)\cdot V_2].$$

This proves (1.5) is surjective.

Let us consider $[\Phi_1]$ and $[\Phi_2]$ in $Q\operatorname{Aut}(G)_{V_1,V_2}$ such that

$$[V_1, \Phi_2 \cdot V_2] = [V_1, \Phi_1 \cdot V_2].$$

We are going to show that $[\Phi_2] = [\Phi_1]$. Since $(V_1, \Phi_2 \cdot V_2)$ and $(V_1, \Phi_1 \cdot V_2)$ share the same orbit, then there exists $(\Psi, \sigma_1, \sigma_2) \in \operatorname{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s$ such that $(V_1, \Phi_2 \cdot V_2) = (\Psi, \sigma_1, \sigma_2) \cdot (V_1, \Phi_1 \cdot V_2)$. Then we have

$$\Psi \cdot V_1 = \sigma_1^{-1} \cdot V_1 \qquad \text{and} \qquad \left(\Phi_1^{-1} \circ \Psi^{-1} \circ \Phi_2\right) \cdot V_2 = \sigma_2 \cdot V_2.$$

Therefore, $\varphi_1 := \Psi \in \mathcal{B}\operatorname{Aut}(G, V_1)$ and $\varphi_2 := \Phi_1^{-1} \circ \Psi^{-1} \circ \Phi_2 \in \mathcal{B}\operatorname{Aut}(G, V_2)$. Finally, we have

$$\Phi_2 = \varphi_1 \circ \Phi_1 \circ \varphi_2,$$

which proves $[\Phi_2] = [\Phi_1]$, and so that (1.5) is injective.

(3) It is an immediate consequence from the definition of the map (1.4).

□

Theorem 1.19 gives not only a perfect enumeration of the families of regular product-quotient surfaces corresponding to an **ordered** pair of topological types of $G$-coverings of the projective line $(C_1, \phi_1)$ and $(C_2, \phi_2)$ but also how to realize these families. Indeed, given $\Psi \in \operatorname{Aut}(G)$, then $(C_1, \phi_1)$ and $(C_2, \phi_2 \circ \Psi^{-1})$ define a product-quotient surface realizing an irreducible family. Theorem 1.19 translates as each family given by topological types of $C_1$ and $C_2$ are obtained in this way via an automorphism of $\operatorname{Aut}(G)$. Furthermore, two automorphisms $\Psi_1$ and $\Psi_2$ define the same family if they are $(V_1, V_2)-$related, or equivalently if their class in $Q\operatorname{Aut}(G)_{V_1,V_2}$ is the same.

Thus, all families may be realized by a pair $(C_1, \phi_1)$ and $(C_2, \phi_2 \circ \Psi^{-1})$ via a automorphism representative $\Psi$ for each class in $Q\operatorname{Aut}(G)_{V_1,V_2}$.

We consider **ordered** pairs of topological types since the Remark 1.12, where we have observed that exchanging $C_1$ and $C_2$ defines an involution on $\bigcup \mathcal{T}^{r,s}(G)$ connecting isomorphic families.

If we are interested in counting the families given by two different topological types of $G$-coverings, then it is sufficient to choose an order of them and then apply Theorem 1.19.

However, to enumerate the families of product-quotient surfaces associated to twice the same topological type we need to study how the exchange of the factors acts on $Q\operatorname{Aut}(G)_{V,V}$.

**Proposition 1.20.** *The exchange of the factors acts on $Q\operatorname{Aut}(G)_{V,V}$ as the involution*

$$Q\operatorname{Aut}(G)_{V,V} \to Q\operatorname{Aut}(G)_{V,V}, \qquad [\Phi] \mapsto [\Phi^{-1}].$$

*Proof.* The exchange of the factors is a map from $\pi^{-1}([V],[V])$ to itself sending each $[V, \Phi \cdot V]$ to $[\Phi \cdot V, V] = [V, \Phi^{-1} \cdot V]$. □

**Corollary 1.21.** *Let $C_1$ and $C_2$ be two $G$-coverings of $\mathbb{P}^1$ and let $V_1$ and $V_2$ be spherical systems of generators of them respectively. Then the cardinality of the set of families of product-quotient surfaces given by the topological types of $C_1$ and $C_2$ is equal to*

(1) *the cardinality of* $Q\,\mathrm{Aut}(G)_{V_1,V_2}$*, if* $C_1$ *and* $C_2$ *are not topological equivalent;*
(2) *the cardinality of* $Q\,\mathrm{Aut}(G)_{V_1,V_1}/\left(\Phi \mapsto \Phi^{-1}\right)$*, if* $C_1$ *and* $C_2$ *are topological equivalent.*

Let us give an example how we use Theorem 1.19 and Corollary 1.21:

**Example 1.22.** Let $G = S_3 \times \mathbb{Z}_3^2$. We are going to compute all regular product-quotient surfaces with quotient model $(C_1 \times C_2)/G$ where the natural fibrations $\lambda_1 \colon C_1 \to \mathbb{P}^1$ and $\lambda_2 \colon C_2 \to \mathbb{P}^1$ are both ramifying over three points.

From Example 1.9, then $C_1$ and $C_2$ are topological equivalent described by the algebraic data

$$V := [(\tau, (1,0)), (\tau\sigma, (0,1)), (\sigma^2, (2,2))].$$

We need to compute the subgroup $\mathcal{B}\,\mathrm{Aut}(G, V) \le \mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)$.

Firstly we note that

$$\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2) \cong \mathrm{Aut}(S_3) \times \mathrm{GL}_2(\mathbb{Z}_3).$$

Hence every element of $\mathcal{B}\,\mathrm{Aut}(G, V)$ can be written as a pair $(\Psi, M)$, where $\Psi \in \mathrm{Aut}(S_3)$, and $M \in \mathrm{GL}_2(\mathbb{Z}_p)$.

The action of $\mathcal{B}_3$ on $[(1,0), (0,1), (2,2)]$ permutes its entries, since $\mathbb{Z}_3^2$ is abelian. Therefore, the automorphisms $M \in \mathrm{GL}_2(\mathbb{Z}_3)$ of braid type on it are those permuting its entries. Such automorphisms belong to the subgroup $\langle M_1, M_2\rangle \cong S_3$ generated by

$$M_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad M_2 := \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}.$$

Let $(\Psi, M)$ be of braid type on $V$, and let $\eta$ be a braid in $\mathcal{B}_3$ such that $(\Psi, M)\cdot V = \eta \cdot V$. We observe that the signature of $V$ is $[6, 6, 9]$: since the third number is different from the other two, and the automorphisms send elements in elements of the same order, then the permutation image of $\eta$ in $S_3$ fix the number three. This implies that $M$ fixes $(2, 2)$, so $M \in \langle M_1\rangle \cong \mathbb{Z}_2$. Therefore,

$$\mathcal{B}\,\mathrm{Aut}(G, V) \le \mathrm{Aut}(S_3) \times \langle M_1\rangle \cong S_3 \times \mathbb{Z}_2.$$

Let us choose two generators of $\mathrm{Aut}(S_3)$: let $\Psi_1$ be the inner automorphism given by $\tau$ and let $\Psi_2$ be the inner automorphism of $\sigma^2$. We observe that $(\Psi_1, \mathrm{Id})$ and $(\Psi_2 \circ \Psi_1, M_1)$ are of braid type on $V$, since they act on $V$ respectively as the braids $\sigma_1\sigma_2^2\sigma_1$ and $\sigma_1$. Since they generate the whole $\mathrm{Aut}(S_3) \times \langle M_1\rangle$ then

$$\mathcal{B}\,\mathrm{Aut}(G, V) = \mathrm{Aut}(S_3) \times \langle M_1\rangle \cong S_3 \times \mathbb{Z}_2.$$

Now we can compute $Q\,\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)_{V,V}$, which as observed is the set of double cosets

$$Q\,\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)_{V,V} = {}_{\mathcal{B}\,\mathrm{Aut}(G,V)}\backslash^{(\mathrm{Aut}(S_3)\times \mathrm{GL}_2(\mathbb{Z}_3))}/_{\mathcal{B}\,\mathrm{Aut}(G,V)}.$$

Since $\mathcal{B}\,\mathrm{Aut}(G, V) = \mathrm{Aut}(S_3) \times \langle M_1\rangle$ contains the subgroup $\mathrm{Aut}(S_3) \times \{1\}$, which is normal in $\mathrm{Aut}(S_3) \times \mathrm{GL}_2(\mathbb{Z}_3)$, then we have the following natural identification

$$Q\,\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)_{V,V} \cong {}_{\langle M_1\rangle}\backslash^{\mathrm{GL}_2(\mathbb{Z}_3)}/_{\langle M_1\rangle}. \tag{1.6}$$

More precisely, the correspondence sends $[(\mathrm{Id}_{S_3}, A)] \leftrightarrow [A]$.

From (1.3) and Theorem 1.19 we can conclude that

$$\mathcal{T}^{3,3}(S_3 \times \mathbb{Z}_p^2) \cong Q\operatorname{Aut}(G)_{V,V} \cong {}_{\langle M_1\rangle}\backslash^{\mathrm{GL}_2(\mathbb{Z}_3)}/_{\langle M_1\rangle}.$$

However, we are majorly interested to find the set of families of product-quotient surfaces given by the pair $V$, $V$. As proved in the Corollary 1.21, it is sufficient to determine

$$Q\operatorname{Aut}(G)_{V_1,V_1} / \left(\Phi \mapsto \Phi^{-1}\right).$$

This is the quotient of $\mathrm{GL}_2(\mathbb{Z}_p)$ by the simultaneous action of the three involutions $A \mapsto M_1 A$, $A \mapsto AM_1$ and $A \mapsto A^{-1}$. These involutions generate a group of order 8 isomorphic to a dihedral group. Hence

$$Q\operatorname{Aut}(G)_{V_1,V_1} / \left(\Phi \mapsto \Phi^{-1}\right) \cong \mathrm{GL}_2(\mathbb{Z}_3)/D_4. \tag{1.7}$$

We have proved that families of regular product-quotient surfaces with quotient model $(C_1 \times C_2)/G$ where $\lambda_1 \colon C_1 \to \mathbb{P}^1$ and $\lambda_2 \colon C_2 \to \mathbb{P}^1$ are both ramifying over three points are in bijection with $\mathrm{GL}_2(\mathbb{Z}_3)/D_4$, a set of cardinality 10. More precisely, these families are realized as follows: we consider two copies $(C_1, \phi)$, $(C_2, \phi)$ of the same curve $(C, \phi)$ defined in the Example 1.9 which is described by the algebraic data $V$. This pair of curves define a product-quotient surface realizing a first family. All the other families are realized by product-quotient surfaces each defined by a pair $(C_1, \phi)$ and $(C_2, \phi \circ (\mathrm{Id}, A^{-1}))$, where $A$ is a representative of a class of $\mathrm{GL}_2(\mathbb{Z}_3)/D_4$.

## 2. Finiteness of the classification problem

In this section we follow step-by-step the same arguments of [8] and generalize the results of [8, Prop. 1.14] by removing the assumption $\chi = 1$ there.
As a consequence of this, we describe an algorithm that produces for any fixed pair of natural integers $K^2$ and $\chi$ all regular product-quotient surfaces $S$ of general type with self-intersection $K_S^2 = K^2$ and Euler characteristic $\chi(\mathcal{O}_S) = \chi$.

Let $C_1$ and $C_2$ be two Riemann surfaces of respective genera $g_1, g_2 \geq 2$ and let $G$ be a finite group acting faithfully on both of them. We consider the diagonal action of $G$ on the product $C_1 \times C_2$, which gives a product-quotient surface $S$, the minimal resolution of singularities of the quotient model $X := (C_1 \times C_2)/G$.

The singular points of the quotient model $X$ are images of points in $C_1 \times C_2$ having non-trivial stabilizer by the diagonal action of $G$. Hence, $X$ has only finitely many singular points which are cyclic quotient singularities.
A cyclic quotient singularity of type $\frac{1}{n}(1,a)$ is the singular point realized as the quotient of $\mathbb{C}^2$ by the action of the diagonal linear isomorphism of eigenvalues $\zeta_n$ and $\zeta_n^a$, with $\gcd(n,a) = 1$.
We can attach to $X$ the so-called *basket* of singularities:

**Definition 2.1.** [8, Def. 1.2] A representation of the basket of singularities of $X$ is a multiset

$$\mathcal{B}(X) := \left\{ \lambda \times \left(\frac{1}{n}(1,a)\right) : X \text{ has exactly } \lambda \text{ singularities of type } \frac{1}{n}(1,a) \right\}.$$

We use the word "representation" since $X$ may have several representatives of its basket, essentially since a singularity of type $\frac{1}{n}(1,a)$ is isomorphic to a singularity of type $\frac{1}{n}(1,a')$, where either $a=a'$ or $aa' \equiv 1 \mod n$.

In [8] the authors used the minimal resolution of a cyclic quotient singularity as *Hirzebruch-Jung string* to compute the correction terms to the self-intersection and the topological characteristic of the product-quotient surface $S$.
We need to remind these these correction terms.

**Definition 2.2.** ([8, Definition 1.5]) Let $x$ be a singularity of type $\frac{1}{n}(1,a)$ with $\gcd(n,a)=1$, and let $1 \le a' < n$ be the inverse of $a$ modulo $n$, $a' = a^{-1}$. Write $\frac{n}{a}$ as a continued fraction

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}} = [b_1, \dots, b_l]$$

We define the following correction terms

- $k_x := k(\frac{1}{n}(1,a)) = -2 + \frac{2+a+a'}{n} + \sum_{i=1}^{l}(b_i - 2) \ge 0$;
- $e_x := e(\frac{1}{n}(1,a)) = l + 1 - \frac{1}{n} \ge 0$;
- $B_x := 2e_x + k_x$.

Let $\mathcal{B}$ be the basket of singularities of $X$. Then we denote by

$$k(\mathcal{B}) := \sum_x k_x, \qquad e(\mathcal{B}) := \sum_x e_x, \qquad B(\mathcal{B}) := \sum_x B_x.$$

**Theorem 2.3.** *([8, Prop. 1.6 and Cor. 1.7]) Let $\rho\colon S \to X$ be the minimal resolution of singularities of $X = (C_1 \times C_2)/G$. Then the self-intersection of the canonical divisor of $S$ and its topological Euler characteristic are equal to*

$$K_S^2 = \frac{8(g_1-1)(g_2-1)}{|G|} - k(\mathcal{B}), \qquad \text{and} \qquad e(S) = \frac{4(g_1-1)(g_2-1)}{|G|} + e(\mathcal{B}).$$

*Furthermore, it holds*

$$K_S^2 = 8\chi(\mathcal{O}_S) - \frac{1}{3}B(\mathcal{B}).$$

From now on we shall restrict to product-quotient surfaces $S$ of general type that are regular, namely $C_i/G \cong \mathbb{P}^1$.
As supposed from the previous Subsection 1.2, we shall describe $S$ in a pure algebraic way by using a pair of spherical systems of generators

$$[g_1, \dots, g_r] \qquad \text{and} \qquad [h_1, \dots, h_s]$$

of the pair of $G$-coverings $C_1$ and $C_2$ of $\mathbb{P}^1$.

**Remark 2.4.** In [8, Subsec. 1.2] is shown how to determine the number of cyclic quotient singularities (and their types) of the quotient model $X = (C_1 \times C_2)/G$ from the algebraic data of a pair of spherical systems of generators.

In this way, we read the basket of singularities of $S$ form the pair $[g_1, \dots, g_r]$ and $[h_1, \dots, h_s]$, and then determine the invariants $K_S^2$ and $\chi(\mathcal{O}_S)$ by using Theorem 2.3.

Finally, we states the preliminaries to extend [8, Prop. 1.14] to any natural integer $\chi$.

**Definition 2.5.** Fix an $r$-tuple of natural numbers $t := [m_1, \dots, m_r]$, and a basket of singularities $\mathcal{B}$. Then we associate to these the following numbers:

$$\Theta(t) := -2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right);$$
$$\alpha(t, \mathcal{B}) := \frac{12\chi + k(\mathcal{B}) - e(\mathcal{B})}{6\Theta(t)}.$$

We recall

**Definition 2.6.** The minimal positive integer $I_x$ such that $I_x K_X$ is Cartier in $x$ is called the *index* of the singularity $x$.

The index of $X$ is the minimal positive integer $I$ such that $I$ is Cartier. In particular, $I = \mathrm{lcm}_{x \in \mathrm{Sing}\, X} I_x$.

It is well known that the index of a cyclic quotient singularity $\frac{1}{n}(1, a)$ is

$$I_x = \frac{n}{\gcd(n, a+1)}.$$

By Lemma [8, Lem. 1.10], fixed a pair of natural integers $(K^2, \chi)$, there are only finitely many basket of singularities $\mathcal{B}$ for which there exists a product-quotient surface $S$ with invariants $K_S^2 = K^2$, $\chi(\mathcal{O}_S) = \chi$, and having a quotient model with a representation of the basket of singularities equal to $\mathcal{B}$.

We need to extend [8, Prop. 1.14] to any natural integer $\chi$ to bound, for fixed $K^2$, $\chi$, and $\mathcal{B}$, the possibilities for

- $|G|$;
- $t_1 := [m_1, \dots, m_r]$,
- $t_2 := [n_1, \dots, n_s]$,

of a product-quotient surface $S$ with $K_S^2 = K^2$, $\chi(S) = \chi$, and basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ equal to $\mathcal{B}$ such that the pair of spherical systems of generators of $C_1$ and $C_2$ have respectively signature $t_1$ and $t_2$.

**Proposition 2.7.** *Fix $(K^2, \chi) \in \mathbb{Z} \times \mathbb{Z}$, and fix a possible basket of singularities $\mathcal{B}$ for $(K^2, \chi)$. Let $S$ be a product-quotient surface $S$ of general type such that*

i. $K_S^2 = K^2$;
ii. $\chi(S) = \chi$;
iii. *the basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ equals $\mathcal{B}$.*

*Then*

a) $g(C_1) = \alpha(t_2, \mathcal{B}) + 1$, $g(C_2) = \alpha(t_1, \mathcal{B}) + 1$;
b) $|G| = \frac{8\alpha(t_1, \mathcal{B})\alpha(t_2, \mathcal{B})}{K^2 + k(\mathcal{B})}$;
c) $r, s \leq \frac{K^2 + k(\mathcal{B})}{2} + 4$;
d) $m_i$ *divides* $2\alpha(t_1, \mathcal{B})I$, $n_j$ *divides* $2\alpha(t_2, \mathcal{B})I$;
e) *there are at most* $|\mathcal{B}|/2$ *indices* $i$ *such that* $m_i$ *does not divide* $\alpha(t_1, \mathcal{B})$, *and similarly for the* $n_j$;

f) $m_i \leq \frac{1+I\frac{K^2+k(\mathcal{B})}{2}}{f(t_1)}$, $n_i \leq \frac{1+I\frac{K^2+k(\mathcal{B})}{2}}{f(t_2)}$, *where* $I$ *is the index of* $X$, *and* $f(t_1) := \max(\frac{1}{6}, \frac{r-3}{2})$, $f(t_2) := \max(\frac{1}{6}, \frac{s-3}{2})$*;*

g) *except for at most* $|\mathcal{B}|/2$ *indices* $i$*, the sharper inequality* $m_i \leq \frac{1+\frac{K^2+k(\mathcal{B})}{4}}{f(t_1)}$ *holds, and similarly for the* $n_j$*.*

**Remark 2.8.** Note that *b*) shows $t_1$ and $t_2$ determines the order of $G$. *c*) and *f*) implies there are only finitely many possibilities for the signatures $t_1, t_2$. Instead, *d*), *e*), and *g*) are strictly necessary to obtain an efficient algorithm.

*Proof.* It is sufficient to prove *a*) since the other points have the same proof as [8]. From Theorem 2.3, then

$$\Theta(t_1)\alpha(t_1,\mathcal{B}) = \frac{1}{2}\frac{24\chi + 2k(\mathcal{B}) - 2e(\mathcal{B})}{6} = \frac{24\chi - B(\mathcal{B}) + 3k(\mathcal{B})}{6}$$
$$= 3\frac{8\chi - \frac{B(\mathcal{B})}{3} + k(\mathcal{B})}{12} = \frac{K^2 + k(\mathcal{B})}{4},$$

and then by the Theorem 2.3 and Hurwitz' formula 1.2, we have

$$\alpha(t_1,\mathcal{B}) = \frac{K^2 + k(\mathcal{B})}{4\Theta(t_1)} = \frac{8(g(C_1)-1)(g(C_2)-1)}{4|G|\left(-2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right)}$$
$$= \frac{8(g(C_1)-1)(g(C_2)-1)}{4(2g(C_1)-2)}.$$

□

2.1. **Description of the classification algorithm.** Fixed a pair $(K^2, \chi) \in \mathbb{N} \times \mathbb{N}$, the next goal is to write a MAGMA script to find all minimal regular surfaces $S$ of general type with $K_S^2 = K^2$, and $\chi(S) = \chi$, which are product-quotient surfaces. A commented version of the MAGMA code is available here:
https://github.com/Fefe9696/PQ_Surfaces_with_fixed_Ksquare_chi

We describe here the strategy, and explain how the most important scripts work. Most of the scripts are modification of those in [8]. Since those scripts were written under the assumption $\chi = 1$, we generalize all of them to allow any value of $\chi$. In the Introduction of the present paper we indicate the other main improvements we did.

We fix a couple $(K^2, \chi)$. Note that by minimality of $S$, and by Theorem 2.3, then $K^2 \in \{1, \ldots, 8\chi\}$, and the case $K^2 = 8\chi$ corresponds to surfaces isogenous to a product.

**Step 1:** The script **Baskets** lists all the *possible basket of singularities* $\mathcal{B}$ for $(K^2, \chi)$. Indeed, there are only finitely many of them by [8, Lem. 1.10]. The input is $B(\mathcal{B}) = 3(8\chi - K^2)$, so to get for instance all baskets for $(K^2, \chi) = (28, 4)$, we need *Basket*(12).

**Step 2:** From the Proposition 2.7 once we know the basket of singularities of $X = (C_1 \times C_2)/G$, then there are finitely many possible signatures of a pair of spherical systems of generators of $C_1$ and $C_2$. **ListOfTypes** computes them using the inequalities in the Proposition 2.7. Here the input is $K^2$, and $\chi$, so *ListOfTypes*

first computes *Baskets(*$3(8\chi - K^2)$*)*, and then computes for each basket all numerically compatible signatures. The output is a list of pairs, the first element of each pair being a basket, and the second element being the list of all signatures compatible with that basket.
**Step 3:** Every surface produces two signatures, one for each curve $C_i$, both compatible with the basket of singularities of $X$; if we know the signatures and the basket, then Proposition 2.7 *b*) tells us the order of $G$. **ListGroups**, whose input is $K^2$, and $\chi$, first computes *ListOfTypes*$(K^2, \chi)$. Then for each pair of signatures in the output, it calculates the order of the group. Next it searches for the groups of given order which admit appropriate spherical systems of generators corresponding to both signatures. Here we use the database in [16] if we are in one of the cases classified there, otherwise we use the function *FindGenerators* developed in the work [16].
For each affirmative answer, it stores the triple (basket, pair of signatures, group) in a list, which is the main output.

The script has some shortcuts:

- Let $t_1$ and $t_2$ be the pair of signatures and let $\mathbb{T}(t_1)$ and $\mathbb{T}(t_2)$ be their respective orbifold groups (see the Remark 1.6). Then the order of the abelianization $G^{ab}$ of $G$ has to divide the order the abelianization of $\mathbb{T}(t_1)$ and $\mathbb{T}(t_2)$:
$$|G^{ab}| \quad \text{divides} \quad |\mathbb{T}(t_1)^{ab}|, |\mathbb{T}(t_2)^{ab}|. \tag{2.1}$$
Indeed, the orbifold (surjective) homomorphisms $\mathbb{T}(t_1) \to G$ and $\mathbb{T}(t_2) \to G$ extend to surjective homomorphisms
$$\mathbb{T}(t_1)^{ab} \to G^{ab}, \qquad \mathbb{T}(t_2)^{ab} \to G^{ab}.$$
Hence *ListGroups* checks first if $G$ satisfies (2.1): if not, this case not occur.
- If the pair of signatures $t_1$ and $t_2$ admits orbifold groups $\mathbb{T}(t_1)$ and $\mathbb{T}(t_2)$ such that the orders of their abelianization are coprime numbers, then $G$ is forced to be a perfect group. This follows directly from the condition (2.1).
MAGMA knows all perfect groups of order $\leq 50000$, and then *ListGroups* checks first if there are perfect groups of the right order: if not, this case can not occur.
- If:
  - either the expected order of the group is 1024 or bigger than 2000, since MAGMA does not have a list of the finite groups of this order;
  - or the order is a number as *e.g.*, 1728, where there are too many isomorphism classes of groups;

  then *ListGroups* just stores these cases in a list, secondary output of the script. These "exceptional" cases have to be considered separately.

**Step 4:** The basket of singularities of a surface described by a couple of spherical systems $[g_1, \dots, g_r]$ and $[h_1, \dots, h_s]$ depends only by the conjugacy classes of $g_i$ and $h_j$, from [17, Rem. 4.7.2]. **ExistingSurfaces** runs on the output of

*ListGroups*$(K^2, \chi)$, and throws away all triples giving rise only to surfaces whose singularities do not correspond to the basket.

**Step 5:** Each triple (basket, pair of signatures, group) in the output *ExistingSurfaces*$(K^2, \chi)$ gives many different pairs of compatible spherical systems of generators. On them there is the action of $\mathrm{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s$ described in Subsection 1.2. Therefore, **FindSurfaces** uses Theorem 1.19 and Corollary 1.21 to pick up only one pair of spherical systems of generators for any family of product-quotient surfaces compatible with the triple (basket, pair of signatures, group). Thus, the output is a list of (basket, sph1, sph2, group), where sph1 and sph2 are spherical systems of group compatible with pair of signatures and basket.

## 3. Classification of regular product-quotient surfaces with $23 \leq K^2 \leq 32$ and $\chi = 4$.

In this section we prove the main Theorem 0.2 presented in the introduction.

We have run the function *FindSurfaces* described in the previous Subsection 2.1 on each triple of the output of *ExistingSurfaces*$(K^2, \chi)$, where $K^2 \in \{23, \dots, 32\}$ and $\chi = 4$. This has given all the families in tables 17 to 29 of the appendix with the only exception of families *no.* 267 and 544, which are the only cases occurred on those skipped by *ListGroups* and stored in its secondary output.

Thus, to prove the main Theorem 0.2 it remains to show that

(1) among all the exceptional cases skipped by *ListGroups*, only two cases occur, which are *no.* 267 and 544;
(2) all the obtained families of tables 17 to 29 are of general type and those on tables 17 to 28 are also minimal.

This will be the content of the next two subsections 3.1 and 3.2.

### 3.1. The exceptional cases.

For each $K^2 \in \{23, \dots, 32\}$, the list of cases skipped by *ListGroups*$(K^2, 4)$ and stored in its secondary output can be found here:
https://github.com/Fefe9696/PQ_Surfaces_with_fixed_Ksquare_chi

The main theorem of this subsection is the following:

**Theorem 3.1.** *There are exactly two groups $G$ admitting an appropriate pair of spherical systems of generators compatible with one of the triples of the secondary output of ListGroups$(K^2, 4)$, for $K^2 \in \{23, \dots, 32\}$:*

| *no.* | $K_S^2$ | *Sing(X)* | $t_1$ | $t_2$ | $G$ | $N$ |
|---|---|---|---|---|---|---|
| 267 | *26* | $1/4, 1/2^2, 3/4$ | $3^2, 4$ | $3^2, 4$ | $G(1944, 3875)$ | *2* |
| 544 | *24* | $1/6, 1/2^2, 5/6$ | $2, 4, 6$ | $2, 6, 8$ | $G(768, 1086051)$ | *2* |

Table 2

A proof of this theorem can be found in the .txt files attached to this paper, one for each $K^2 \in \{23, \dots, 32\}$. More precisely, in these files we provide a step-by-step proof of how to exclude the cases omitted by *ListGroups* until we encounter the only two cases above that actually occur.

However, to illustrate the main strategy we have employed to exclude these cases, here we only discuss those with $K^2 = 32$, which already consist of a significant list of 152 cases. Therefore, we need to prove

**Theorem 3.2.** *No one of the cases skipped by ListGroups$(32, 4)$ gives a product-quotient surface $S$ with $K_S^2 = 32$ and $\chi(\mathcal{O}_S) = 4$.*

*Proof.* It follows from propositions 3.5, 3.9, 3.11, 3.15, and 3.16 below. □

The rest of this section is devoted to give a proof of the series of propositions used to prove Theorem 3.2.

We use two MAGMA functions to prove these propositions and more in general Theorem 3.1:

- **HowToExclude** takes in input a list of triples as those of the second output of *ListGroups* that have an order of the group different from 1024 and less or equal to 2000. For each triple (basket, $(t_1, t_2)$, $ord$) of the list it returns those groups with order $ord$ admitting a pair of spherical systems of generators of signatures $t_1$ and $t_2$. This function uses such as *ListGroups* the database and function *FindGenerators* in [16];
- The function **HowToExcludePG** works similarly such as *HowToExclude*. Hence, it takes in input a list of triples (basket, $(t_1, t_2)$, $ord$), where $ord$ is $\leq 50000$, and returns those groups with order $ord$ that are perfect and admit a pair of spherical systems of generators of signatures $t_1$ and $t_2$.

We also need the following

**Proposition 3.3.** *Let $G$ be a finite group that admits a spherical system of generators of signature $[a_1, a_2, a_3, b_1, \dots, b_k]$. Let us suppose $G$ have a normal subgroup $H$ of index a prime number $p \geq 2$ and that $p$ does not divide $b_1, \dots, b_k$. Then*

- *if $p$ does not divide only one among $a_1, a_2, a_3$, e.g. $p \nmid a_3$, then $H$ admits a spherical system of generators of signature $[a_1/p, a_2/p, a_3^p, b_1^p, \dots, b_k^p]$;*
- *if $p$ divides each of $a_1, a_2, a_3$, then $H$ admits either a spherical system of generators having one of the following signatures:*
  *(1) $[a_1/p, a_2/p, a_3^p, b_1^p, \dots, b_k^p]$;*
  *(2) $[a_1/p, a_2^p, a_3/p, b_1^p, \dots, b_k^p]$;*
  *(3) $[a_1^p, a_2/p, a_3/p, b_1^p, \dots, b_k^p]$;*
  *or, if $p \neq 2$, it admits a generating vector of type*

$$\left[\frac{p-1}{2}; a_1/p, a_2/p, a_3/p, b_1^p, \dots, b_k^p\right].$$

*In other words, it there exists a $H$-covering of a curve of genus $\frac{p-1}{2}$ whose branch locus has ramification indices $a_1/p, a_2/p, a_3/p, b_1^p, \dots, b_k^p$.*

*Proof.* By assumption, $G$ has a spherical system of generators $[g_1, g_2, g_3, h_1 \dots, h_k]$ which defines a $G$-covering $C \to \mathbb{P}^1$ whose branch locus $v_1, v_2, v_3, q_1, \dots, q_k \in \mathbb{P}^1$

has ramification indices $a_1, a_2, a_3, b_1, \dots, b_k$ respectively. Furthermore, the existence of a normal subgroup $H$ of index $p$ gives the following triangular commutative diagram:

$$\begin{array}{ccc} C & & \\ {\scriptstyle /H}\downarrow & \searrow^{/G} & \\ C/H & \xrightarrow[/\mathbb{Z}_p]{} & \mathbb{P}^1. \end{array}$$

Note that $h_i \in H$ since $h_iH$ has order in $G/H \cong \mathbb{Z}_p$ that divides both $p$ and the order $b_i$ of $h_i$. Hence $q_1, \dots, q_k$ are not in the branch locus of $C/H \to \mathbb{P}^1$, which has then to ramify over at most $r \leq 3$ points with ramification indices $p$.
By Hurwitz formula (1.2), we get

$$2g(C/H) - 2 = p\left(-2 + r\frac{p-1}{p}\right) \implies g(C/H) = \frac{p-1}{2}(r-2). \tag{3.1}$$

Hence $r$ is forced to be either equal to 2 or 3. If $r = 2$, then $C/H \cong \mathbb{P}^1$, and we can assume without lost of generalities that $v_3$ is not in the branch locus, so in other words $g_3 \in H$.
We want to determine the signature of a spherical system of generators that defines $C \to C/H \cong \mathbb{P}^1$. Each point of the fibre of $q_i$ via $C/H \to \mathbb{P}^1$ is contained in the branch locus of $C \to C/H$ and has ramification index $b_i$, since $h_i \in H$. Note that the cardinality of the fibre is exactly $p$ for these points $q_i$. The same holds for $v_3$, since also $g_3$ belongs to $H$.
Instead, the fibre of $v_i$ on $C/H$ consists of only one point, $i = 1, 2$. The ramification index of this point for $C \to C/H$ equals the order of $\langle g_i \rangle \cap H$, which is $a_i/p$. We therefore obtain the signature $[a_1/p, a_2/p, a_3^p, b_1^p, \dots, b_k^p]$.

The case $r = 3$ can be discussed by using the same argument. □

**Remark 3.4.** Since product-quotient surfaces with $K_S^2 = 32$ and $\chi(\mathcal{O}_S) = 4$ are surfaces isogenous to a product, then their basket is always empty. For this reason, we shall avoid repeating which is the basket of the triples of the cases skipped by *ListGroups*.

A first result is the following

**Proposition 3.5.** *There are exactly five groups $G$ of order different from 1024 and less or equal than 2000 admitting an appropriate pair of spherical systems of generators compatible with one of the triples of the secondary output of ListGroups:*

| $t_1$ | $t_2$ | $G$ |
|---|---|---|
| $2,4,6$ | $2^3,4$ | $G(768,1086051)$ |
| $2,4,6$ | $2^3,4$ | $G(768,1086052)$ |
| $2,4,6$ | $2,4,20$ | $G(960,5719)$ |
| $2,4,6$ | $2,4,12$ | $G(1152,157849)$ |
| $2,4,5$ | $2,4,12$ | $G(1920,240996)$ |

*However, no one of these cases gives product-quotient surfaces isogenous to a product.*

*Proof.* We select in a list those triples of the secondary output of *ListGroups* having a order of the group different from 1024 and less or equal to 2000. Then we run *HowToExclude* on this list and we obtain the above table.
However, we use *ExistingSurfaces* for each of the rows of the table to check that no one gives a product-quotient surface isogenous to a product. □

As a consequence of the previous lemma, it remains to discuss only 65 of 152 cases skipped by *ListGroups*, that are those of Tables 3 and 4 below.

| *no.* | $t_1$ | $t_2$ | $\|G\|$ |
|---|---|---|---|
| 1 | 2, 3, 8 | 2, 5² | 3840 |
| 2 | 2, 3, 7 | 4, 4, 4 | 2688 |
| 3 | 2, 3, 7 | 2, 3, 18 | 6048 |
| 4 | 2, 3, 7 | 2, 4, 8 | 5376 |
| 5 | 2, 3, 7 | 3, 3, 5 | 5040 |
| 6 | 2, 3, 7 | 2, 5, 6 | 5040 |
| 7 | 2, 3, 7 | 2, 8, 8 | 2688 |
| 8 | 2, 3, 7 | 3, 3, 15 | 2520 |
| 9 | 2, 3, 7 | 2, 3, 7 | 28224 |
| 10 | 2, 3, 7 | 2, 5, 30 | 2520 |
| 11 | 2, 3, 7 | 2, 3, 10 | 10080 |
| 12 | 2, 3, 7 | 2, 2, 2, 4 | 2688 |
| 13 | 2, 3, 7 | 2, 6, 15 | 2520 |
| 14 | 2, 3, 7 | 3, 5, 5 | 2520 |
| 15 | 2, 3, 7 | 2, 3, 30 | 5040 |
| 16 | 2, 4, 5 | 3, 3, 4 | 3840 |
| 17 | 2, 3, 9 | 2, 4, 5 | 5760 |
| 18 | 2, 3, 9 | 2, 5, 6 | 2160 |
| 19 | 2, 3, 12 | 2, 4, 6 | 2304 |
| 20 | 2, 3, 10 | 2, 4, 6 | 2880 |
| 21 | 2, 3, 8 | 2, 4, 12 | 2304 |
| 22 | 2, 3, 8 | 2, 5, 6 | 2880 |
| 23 | 2, 3, 22 | 2, 4, 5 | 2640 |
| 24 | 2, 3, 12 | 2, 4, 5 | 3840 |
| 25 | 2, 3, 14 | 2, 4, 6 | 2016 |
| 26 | 2, 3, 8 | 2, 4, 6 | 4608 |
| 27 | 2, 3, 18 | 2, 4, 5 | 2880 |
| 28 | 2, 3, 10 | 2, 4, 5 | 4800 |
| 29 | 2, 3, 54 | 2, 4, 5 | 2160 |
| 30 | 2, 4, 5 | 2, 4, 6 | 3840 |
| 31 | 2, 3, 30 | 2, 4, 5 | 2400 |
| 32 | 2, 4, 5 | 2, 4, 8 | 2560 |
| 33 | 2, 3, 8 | 2, 4, 5 | 7680 |

TABLE 3

| *no.* | $t_1$ | $t_2$ | $\|G\|$ |
|---|---|---|---|
| 34 | 2, 3, 14 | 2, 4, 5 | 3360 |
| 35 | 2, 3, 8 | 2, 4, 8 | 3072 |
| 36 | 2, 3, 8 | 2, 6, 7 | 2016 |
| 37 | 2, 3, 10 | 2, 3, 10 | 3600 |
| 38 | 2, 3, 8 | 2, 3, 18 | 3456 |
| 39 | 2, 3, 8 | 2, 3, 54 | 2592 |
| 40 | 2, 4, 5 | 2, 5, 6 | 2400 |
| 41 | 2, 3, 8 | 2, 3, 22 | 3168 |
| 42 | 2, 3, 12 | 2, 3, 14 | 2016 |
| 43 | 2, 3, 8 | 2, 3, 30 | 2880 |
| 44 | 2, 3, 8 | 2, 2, 2, 3 | 2304 |
| 45 | 2, 3, 8 | 2, 6, 6 | 2304 |
| 46 | 2, 3, 8 | 3, 4, 4 | 2304 |
| 47 | 2, 3, 10 | 2, 3, 18 | 2160 |
| 48 | 2, 3, 10 | 2, 3, 14 | 2520 |
| 49 | 2, 3, 10 | 2, 3, 12 | 2880 |
| 50 | 2, 3, 8 | 2, 3, 14 | 4032 |
| 51 | 2, 3, 8 | 2, 3, 8 | 9216 |
| 52 | 2, 4, 5 | 2, 4, 5 | 6400 |
| 53 | 2, 3, 8 | 2, 3, 12 | 4608 |
| 54 | 2, 3, 8 | 2, 3, 10 | 5760 |
| 55 | 2, 3, 9 | 3, 3, 5 | 2160 |
| 56 | 2, 3, 9 | 2, 3, 12 | 3456 |
| 57 | 2, 3, 12 | 3, 3, 4 | 2304 |
| 58 | 2, 3, 9 | 2, 3, 18 | 2592 |
| 59 | 2, 3, 9 | 2, 3, 30 | 2160 |
| 60 | 2, 3, 9 | 2, 3, 9 | 5184 |
| 61 | 3, 3, 4 | 3, 3, 4 | 2304 |
| 62 | 2, 3, 9 | 3, 3, 4 | 3456 |
| 63 | 2, 4, 6 | 2, 4, 6 | 2304 |
| 64 | 2, 3, 12 | 2, 3, 12 | 2304 |
| 65 | 2, 4, 8 | 2, 4, 8 | 1024 |

TABLE 4

**Remark 3.6.** We remind that as observed in (2.1) the abelianization $G^{ab}$ of $G$ is a quotient of both the abelianizations of the orbifold groups $\mathbb{T}(t_1)$ and $\mathbb{T}(t_2)$. In particular, the order of $G^{ab}$ divides the greatest common divisor of the orders of $\mathbb{T}(t_1)^{ab}$ and $\mathbb{T}(t_2)^{ab}$.

**Remark 3.7.** From Remark 3.6, we automatically get that groups $G$ having group order and a pair of spherical system of generators compatible with one of the triples of Tables 3 and 4

(1) from *no*.1 to *no*.18 are perfect groups;
(2) from *no*.19 to *no*.54 are either perfect groups or $G^{ab} \cong \mathbb{Z}_2$;
(3) from *no*.55 to *no*.62 are either perfect groups or $G^{ab} \cong \mathbb{Z}_3$;
(4) *no*.63 are either perfect groups or $G^{ab}$ is isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$;
(5) *no*.64 are either perfect or $G^{ab}$ is isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_3$ or to $\mathbb{Z}_6$;
(6) *no*.65 are either perfect or $G^{ab}$ is isomorphic to one among $\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$,$\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$;

**Lemma 3.8.** *There are no perfect groups $G$ having group order and a pair of spherical systems of generators of signatures compatible with one of the triples of Tables 3 and 4.*

*Proof.* We use *HowToExcludePG* on the list of triples of Tables 3 and 4 to check that there are no perfect groups having compatible algebraic data. □

**Proposition 3.9.** *There are no groups $G$ having group order and a pair of spherical systems of generators of signatures compatible with one of the triples of Tables 3 and 4 from no.1 to no.18.*

*Proof.* From Remark 3.7 and Lemma 3.8 we can automatically exclude triples of Table 3 from *no*.1 to *no*.18. □

**Remark 3.10.** We need the following classical remarks of group theory:

(1) Let $G$ be a finite group having a normal subgroup $H$ of index a prime number $p \geq 2$. If there is a element $g \notin H$ of order $p$, then
$$0 \to H \to G \to \mathbb{Z}_p \to 0$$
is a splitting exact sequence via the homomorphism section sending $\overline{1} \in \mathbb{Z}_p$ to $g$. In other words, $G = H \rtimes_\phi \mathbb{Z}_p$, where $\phi$ is an automorphism of $H$ of order $p$;
(2) Let $\pi\colon G \to Z$ be a surjective group homomorphism. If $Z$ admits a normal subgroup $T$ of index $k \in \mathbb{N}$, then $H := \pi^{-1}(T)$ is a normal subgroup of $G$ of index $k$. More precisely, $G/H \cong Z/T$.

**Proposition 3.11.** *There are no groups $G$ having group order and a pair of spherical systems of generators defining a product-quotient surface isgenous to a product and compatible with one of the triples from no.19 to no.62 of Tables 3 and 4.*

*Proof.* From Remark 3.7 and Lemma 3.8, groups $G$ from *no*.19 to *no*.62 of Tables 3 and 4 have a commutator subgroup $G' := [G, G]$ of index either equal to 2 or 3.

Hence we can apply Proposition 3.3 to $H = G'$ and say that $G'$ has group order and a pair of spherical systems of generators compatible with one of the triples of Tables 5 and 6.

| $no.$ | $t_1$ | $t_2$ | $\lvert G'\rvert$ |
|---|---|---|---|
| $19(a)$ | $3,3,6$ | $2,2,2,3$ | 1152 |
| (b) | $3,3,6$ | $3,4,4$ | 1152 |
| (c) | $3,3,6$ | $2,6,6$ | 1152 |
| $20(a)$ | $3,3,5$ | $2,2,2,3$ | 1440 |
| (b) | $3,3,5$ | $3,4,4$ | 1440 |
| (c) | $3,3,5$ | $2,6,6$ | 1440 |
| $21(a)$ | $3,3,4$ | $2,2,2,6$ | 1152 |
| (b) | $3,3,4$ | $4,4,6$ | 1152 |
| (c) | $3,3,4$ | $2,12,12$ | 1152 |
| 22 | $3,3,4$ | $3,5,5$ | 1440 |
| 23 | $3,3,11$ | $2,5,5$ | 1320 |
| 24 | $3,3,6$ | $2,5,5$ | 1920 |
| $25(a)$ | $3,3,7$ | $2,2,2,3$ | 1008 |
| (b) | $3,3,7$ | $3,4,4$ | 1008 |
| (c) | $3,3,7$ | $2,6,6$ | 1008 |
| $26(a)$ | $3,3,4$ | $2,2,2,3$ | 2304 |
| (b) | $3,3,4$ | $3,4,4$ | 2304 |
| (c) | $3,3,4$ | $2,6,6$ | 2304 |
| 27 | $3,3,9$ | $2,5,5$ | 1440 |
| 28 | $3,3,5$ | $2,5,5$ | 2400 |
| 29 | $3,3,27$ | $2,5,5$ | 1080 |
| $30(a)$ | $2,5,5$ | $2,2,2,3$ | 1920 |
| (b) | $2,5,5$ | $3,4,4$ | 1920 |
| (c) | $2,5,5$ | $2,6,6$ | 1920 |
| 31 | $3,3,15$ | $2,5,5$ | 1200 |
| $32(a)$ | $2,5,5$ | $2,2,2,4$ | 1280 |
| (b) | $2,5,5$ | $4,4,4$ | 1280 |
| (c) | $2,5,5$ | $2,8,8$ | 1280 |
| 33 | $3,3,4$ | $2,5,5$ | 3840 |
| 34 | $3,3,7$ | $2,5,5$ | 1680 |

TABLE 5

| $no.$ | $t_1$ | $t_2$ | $\lvert G'\rvert$ |
|---|---|---|---|
| $35(a)$ | $3,3,4$ | $2,2,2,4$ | 1536 |
| (b) | $3,3,4$ | $4,4,4$ | 1536 |
| (c) | $3,3,4$ | $2,8,8$ | 1536 |
| 36 | $3,3,4$ | $3,7,7$ | 1008 |
| 37 | $3,3,5$ | $3,3,5$ | 1800 |
| 38 | $3,3,4$ | $3,3,9$ | 1728 |
| 39 | $3,3,4$ | $3,3,27$ | 1296 |
| 40 | $2,5,5$ | $3,5,5$ | 1200 |
| 41 | $3,3,4$ | $3,3,11$ | 1584 |
| 42 | $3,3,6$ | $3,3,7$ | 1008 |
| 43 | $3,3,4$ | $3,3,15$ | 1440 |
| 44 | $3,3,4$ | $2,2,3,3$ | 1152 |
| $45(a)$ | $3,3,4$ | $2,2,3,3$ | 1152 |
| (b) | $3,3,4$ | $3,6,6$ | 1152 |
| 46 | $3,3,4$ | $2,2,3,3$ | 1152 |
| 47 | $3,3,5$ | $3,3,9$ | 1080 |
| 48 | $3,3,5$ | $3,3,7$ | 1260 |
| 49 | $3,3,5$ | $3,3,6$ | 1440 |
| 50 | $3,3,4$ | $3,3,7$ | 2016 |
| 51 | $3,3,4$ | $3,3,4$ | 4608 |
| 52 | $2,5,5$ | $2,5,5$ | 3200 |
| 53 | $3,3,4$ | $3,3,6$ | 2304 |
| 54 | $3,3,4$ | $3,3,5$ | 2880 |
| 55 | $2,2,2,3$ | $5,5,5$ | 720 |
| 56 | $2,2,2,3$ | $2,2,2,4$ | 1152 |
| 57 | $2,2,2,4$ | $4,4,4$ | 768 |
| 58 | $2,2,2,3$ | $2,2,2,6$ | 864 |
| 59 | $2,2,2,3$ | $2,2,2,10$ | 720 |
| 60 | $2,2,2,3$ | $2,2,2,3$ | 1728 |
| 61 | $4,4,4$ | $4,4,4$ | 768 |
| 62 | $2,2,2,3$ | $4,4,4$ | 1152 |

TABLE 6

**Remark 3.12.** We run *HowToExcludePG* on the list of Tables 5 and 6 to see that there are no compatible perfect groups $G'$.

from $no.19$ to $no.36$, and $no.55$

From Remark 3.6, we see that triples of Tables 5 and 6 from $no.19$ to $no.36$ (with the exception of $no.19(c), 20(c), 21(c), 25(c), 26(c)$) together with $no.55$ have $G'$

forced to be a perfect group, which is a contradiction with Remark 3.12.
We run $HowToExclude$ on $no.19(c), 20(c), 21(c), 25(c)$ to prove that there are no groups compatible with those algebraic data.
Instead, we exclude $26(c)$ using the following

**Remark 3.13.** From [2, Lemma 4.11], there are no groups of order 768 having a spherical system of generators of signature $[4, 4, 4]$.

Indeed, we would get $G'' = [G', G']$ of $26(c)$ is a group of order 768 and from Proposition 3.3 it should admit a spherical system of generators of signature $[4, 4, 4]$.

We have excluded all cases from $no.19$ to $no.36$ together with $no.55$ of Tables 3 and 4.

$\underline{no.53, 57, 61}$
Here $no.53$ of Table 4 can be excluded by using Remark 3.13 and the same argument of $26(c)$ applied to $no.53$ of Table 6. We also automatically exclude $no.57$ and $no.61$ of Table 4 by using Remark 3.13 applied to $no.57$ and $no.61$ to Table 6.

$\underline{\text{from } no.37 \text{ to } no.49, \text{ and } no.56, 58, 59, 60, 62}$
We run $HowToExclude$ on the corresponding triples of Table 6 to see that there are only 10 groups, those of Table 7, compatible with the algebraic data. However,

| $no.$ | $t_1$ | $t_2$ | $G'$ |
|---|---|---|---|
| 58 | $2, 2, 2, 3$ | $2, 2, 2, 6$ | $G(864, 2225)$ |
| 58 | $2, 2, 2, 3$ | $2, 2, 2, 6$ | $G(864, 4175)$ |
| 59 | $2, 2, 2, 3$ | $2, 2, 2, 10$ | $G(720, 764)$ |
| 59 | $2, 2, 2, 3$ | $2, 2, 2, 10$ | $G(720, 771)$ |
| 60 | $2, 2, 2, 3$ | $2, 2, 2, 3$ | $G(1728, 12317)$ |
| 60 | $2, 2, 2, 3$ | $2, 2, 2, 3$ | $G(1728, 32133)$ |
| 60 | $2, 2, 2, 3$ | $2, 2, 2, 3$ | $G(1728, 46099)$ |
| 60 | $2, 2, 2, 3$ | $2, 2, 2, 3$ | $G(1728, 46119)$ |
| 60 | $2, 2, 2, 3$ | $2, 2, 2, 3$ | $G(1728, 47853)$ |
| 60 | $2, 2, 2, 3$ | $2, 2, 2, 3$ | $G(1728, 47900)$ |

TABLE 7

we run $ExistingSurfaces$ on this list to cheek that there are not surfaces isogenous to a product.

For the remaining four cases $\underline{no.50, 51, 52, 54}$ we remind Remark 3.12 and so we apply Proposition 3.3 to the commutator $G'' \triangleleft G'$, which has then to admit one of the algebraic data of Table 8. We run $HowToExcludePG$ on this list to automatically exclude $no.50$ and $no.54$ of Table 4.

$\underline{no.52}$
We run $HowToExclude$ and then $ExistingSurfaces$ for $no.52$ of Table 8 to see that there are only four groups $G''(640, n)$ having a pair of spherical system of generators defining a product-quotient surface isogenous to product, where $n =$

| *no.* | $t_1$ | $t_2$ | $\lvert G''\rvert$ |
|---|---|---|---|
| 50 | $4,4,4$ | $7,7,7$ | 672 |
| 51 | $4,4,4$ | $4,4,4$ | 1536 |
| 52 | $2^5$ | $2^5$ | 640 |
| 54 | $4,4,4$ | $5,5,5$ | 960 |

TABLE 8

$7665, 8697, 12278, 15814$.
However, we remind that $G''$ has index 5 in $G'$, which admits a spherical system of generators $[g_1, g_2, g_3]$ of signature $[2, 5, 5]$. Then $g_2 \notin G''$ and it has order 5. This means from Remark 3.10(1) that

$$0 \to G'' \to G' \to \mathbb{Z}_5 \to 0$$

is a splitting exact sequence, so $G' = G'' \rtimes_\phi \mathbb{Z}_5$ trough an automorphism $\phi$ of $G''$ of order 5. We easily check that each of the obtained groups $G''(640, n)$ admits exactly four automorphisms of order 5. However, for each of these automorphisms $\phi$ the semidirect product $G''(640, n) \rtimes_\phi \mathbb{Z}_5$ has abelianization $\mathbb{Z}_2^4 \times \mathbb{Z}_5$, so no one of these groups can be $G'$ of *no*.52 in Table 6, which has abelianization $\mathbb{Z}_5$.
This then excludes groups $G$ of *no*.52 of Table 4.

<u>*no*.51</u>

It remains to only discuss *no*.51 of Table 8. We run *ExSphSyst* to each group of order 1536 to observe that

**Remark 3.14.** There are no groups of order 1536 which admit a spherical system of generators of signature $[4, 4, 4]$.

□

**Proposition 3.15.** *There are no groups $G$ having group order and a pair of spherical systems of generators defining a product-quotient surface isogenous to a product and compatible with one of the triples no.*63 *and no.*64 *of Table 4.*

*Proof.* From remarks 3.7, 3.8, and 3.10(2), then $G$ of *no*.63 admits a normal subgroup $H$ of index 2, whilst $G$ of *no*.64 admits a normal subgroup $H$ of index either 2 of 3. We apply Proposition 3.3 to $H$, which has then one of the following algebraic data:

| *no.* | $t_1$ | $t_2$ | $\lvert H\rvert$ |
|---|---|---|---|
| 63 | $2,2,2,3$ | $2,2,2,3$ | 1152 |
| 63 | $2,2,2,3$ | $3,4,4$ | 1152 |
| 63 | $2,2,2,3$ | $2,6,6$ | 1152 |
| 63 | $3,4,4$ | $3,4,4$ | 1152 |

TABLE 9

| *no.* | $t_1$ | $t_2$ | $\lvert H\rvert$ |
|---|---|---|---|
| 63 | $3,4,4$ | $2,6,6$ | 1152 |
| 63 | $2,6,6$ | $2,6,6$ | 1152 |
| 64 | $3,3,6$ | $3,3,6$ | 1152 |
| 64 | $2,2,2,4$ | $2,2,2,4$ | 768 |

TABLE 10

We run $HowToExclude$ on this list to see that there are 90 groups compatible with one of the algebraic data. However, we then run $ExistingSurfaces$ to check that there are no surfaces isogenous to a product. □

**Proposition 3.16.** *There are no groups $G$ having group order and a pair of spherical systems of generators defining a product-quotient surface isogenous to a product and compatible with the triple no.65 of Table 4.*

*Proof.* From remarks 3.7, 3.8, and 3.10(2), then $G$ of $no.65$ admits a normal subgroup $H$ of index 2. We apply Proposition 3.3 to $H$, which has then one of the algebraic data of Table 11. We run $HowToExclude$ and then $ExistingSurfaces$

| $no.$ | $t_1$ | $t_2$ | $\|H\|$ |
|---|---|---|---|
| 65 | $2,2,2,4$ | $2,2,2,4$ | 512 |
| 65 | $2,2,2,4$ | $4,4,4$ | 512 |
| 65 | $2,2,2,4$ | $2,8,8$ | 512 |
| 65 | $4,4,4$ | $4,4,4$ | 512 |
| 65 | $4,4,4$ | $2,8,8$ | 512 |
| 65 | $2,8,8$ | $2,8,8$ | 512 |

TABLE 11

on Table 11 to see that there are only three groups $H(512, n)$ having a pair of spherical system of generators of signature $[4, 4, 4]$ defining a product-quotient surface isogenous to a product, where $n = 325, 335, 351$.
Since $G$ admits a spherical system of generators $[g_1, g_2, g_3]$ of signature $[2, 4, 8]$ and $H(512, n)$ has signature $[4, 4, 4]$, then $g_1 \notin H(512, n)$, so that from Remark 3.10(1)

$$0 \to H(512, n) \to G \to \mathbb{Z}_2 \to 0$$

is a splitting exact sequence. In other words, $G = H(512, n) \rtimes_\phi \mathbb{Z}_2$ via an automorphism $\phi$ of order 2. For each of these three groups $H(512, n)$, there are respectively 128,384,128 automorphisms $\phi$ of order two such that $H(512, n) \rtimes_\phi \mathbb{Z}_2$ has abelianization that is a quotient of $\mathbb{Z}_4^2$. In particular, for each of of these automorphisms the abelianization is $\mathbb{Z}_2 \times \mathbb{Z}_4$. Furthermore, each of these groups admits a spherical system of generators of signature $[4, 4, 4]$.
However, we run $ExisitngSurfaces$ to prove that no one of them gives a product-quotient surface isogenous to a product. □

3.2. **Rational (-1) curves on product-quotient surfaces.** In this short subsection we investigate which surfaces among those obtained in Theorem 0.2 do not contain $(-1)$-curves, namely smooth rational curves with self-intersection $-1$. First of all we observe that

**Remark 3.17.** All surfaces $S$ obtained in Theorem 0.2 are surfaces of general type. Indeed, from Enriques-Kodaira classification of complex algebraic surfaces, if $q(S)$ is zero, then either $S$ is rational, of $S$ is of general type, or $K_S^2 \le 0$. Therefore, since surfaces of Theorem 0.2 have $K_S^2 \ge 23$, $q(S) = 0$, and $p_g(S) = 3 \ne 0$, then they are of general type.

**Proposition 3.18.** [7, Lemma 6.9] *Let $S$ be a product-quotient surface of general type of quotient model $X$. Assume that the exceptional locus of the minimal resolution of singularities $\rho\colon S \to X$ consists of*

i) *curves of self-intersection $(-3)$ and $(-2)$, or*
ii) *at most two smooth rational curves of self-intersection $(-3)$ or $(-4)$, and $(-2)$-curves.*

*Then $S$ is minimal, so it does not contain $(-1)$-curves.*

**Corollary 3.19.** *Let $S$ be a product-quotient surface belonging to one of the families of tables 17 to 28 of Theorem 0.2. Then $S$ is a minimal surface.*

*Proof.* For each case of tables 17 to 28 (with the exception of *no.*186 to 196) the exceptional curves arising from the basket of singularities of the quotient model $X$ are either of type $i)$ or $ii)$ of Proposition 3.18, so that $S$ is minimal.
Regarding the remain cases *no.* 186 to 196, their basket of singularities is always equal to $\{1/5, 4/5\}$, so the minimality follows directly by [8, Prop. 4.7(3)]. □

## 4. The degree of the canonical map of product-quotient surfaces

In this section we investigate the degree of the canonical map of product-quotient surfaces, with a particular focus to those having geometric genus three.

We briefly explain the strategy and the content of each subsection but first we give the following

**Remark 4.1.** The degree of the canonical map of product-quotient surfaces is bounded from above by 32. Indeed, product-quotient surfaces satisfy the inequality $K_S^2 \leq 8\chi(\mathcal{O}_S)$, see Theorem 2.3, and so replacing Bogomolov-Miyaoka-Yau inequality with $K_S^2 \leq 8\chi(\mathcal{O}_S)$ in the proof of [9, Prop. 4.1], we get $\deg(\Phi_S) \leq 8\chi(\mathcal{O}_S)/\left(\chi(\mathcal{O}_S) - 3\right) \leq 32$.

Let us consider a product-quotient surface $S$ given by a pair of curves $C_1$ and $C_2$ and a finite group $G$ acting (faithfully) on both of them.

The diagonal action of $G$ on the product $C_1 \times C_2$ induces a representation of $G$ on the spaces of 2-forms of $C_1 \times C_2$. Let us denote by $|K_{C_1\times C_2}|^G$ the linear subsystem of the canonical linear system of $C_1 \times C_2$ given by the subspace $H^{2,0}(C_1 \times C_2)^G$ of the $G$-invariant 2-forms.

In Subsection 4.4 we show the relationship between the degree of the canonical map of $S$ and the (schematic) base locus of $|K_{C_1\times C_2}|^G$. Indeed, it holds

$$\deg(\Phi_S) = \frac{1}{|G| \cdot \deg(\Sigma)} \cdot \widehat{M}^2, \tag{4.1}$$

where $\Sigma$ is the image of the canonical map of $S$, and $\widehat{M}$ is the base-point free linear system obtained blowing-up the base locus of $|K_{C_1\times C_2}|^G$.
Note that whenever $p_g(S) = 3$, then the image of the canonical map is $\mathbb{P}^2$, a surface of degree 1, and so the knowledge of the base locus of $|K_{C_1\times C_2}|^G$ is enough to compute $\deg(\Phi_S)$ automatically by using formula (4.1).

The strategy to investigate the base locus of $|K_{C_1\times C_2}|^G$ is the following. The action of $G$ induces a representation on the space of 1-forms $H^{1,0}(C_i)$ via pullback, called in literature *canonical representation*. By standard representation theory, the space of 1-forms splits as a direct sum of isotypic components $H^{1,0}(C_i)^\chi$, $\chi \in Irr(G)$ irreducible character of $G$. The irreducible characters $\chi$ occurring in the character $\chi_{can}$ of the canonical representation are explicitly computable by the Chevalley-Weil formula, see [24, Subsection 1.3].

As a consequence of this, the space of invariant 2-forms $H^{2,0}(C_1 \times C_2)^G$ splits as a direct sum of invariant subspaces $\left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G$, $\chi \in Irr(G)$. Therefore, the base locus of $|K_{C_1\times C_2}|^G$ is simply the intersection of the base loci of such invariant subspaces and then a computation of them solves the problem.

Let us consider the natural inclusion

$$\left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G \subseteq H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}. \tag{4.2}$$

The main Theorem 4.22 of Subsection 4.3 computes the base locus of the linear subsystem of the bigger subspace $H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$, which is discovered to being pure in codimension 1 and union of fibres (with multiplicities) for the natural projections $C_1 \times C_2 \to C_i$, $i = 1, 2$.
The formula to compute explicitly these fibres and their multiplicities is given through Theorem 4.12 in Subsection 4.1. This theorem provides the base locus of the subsystem of the canonical system of a Riemann surface $C$ given by an isotypic component $H^{1,0}(C)^\chi$ of the action of a finite group $G$ on $C$.

Notice that, whenever $\chi$ is of degree one, then (4.2) is an equality. Thus, if $S$ satisfies Property (#) mentioned at the introduction for which

$$\left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G \neq 0 \implies \deg(\chi) = 1$$

for any irreducible character $\chi$, then we know $|K_{C_1\times C_2}|^G$ is spanned by $p_g$ divisors which decomposes as union of fibres for the natural projections $C_1 \times C_2 \to C_i$, $i = 1, 2$. Since two fibres either do not intersect or they intersect transversally at a point, this makes the base locus of $|K_{C_1\times C_2}|^G$ explicit.

**Remark 4.2.** Observe that Property (#) always holds for $G$ abelian group, and it is sometimes satisfied for other non-abelian groups, since we are only interested to those characters of $G$ for which (4.2) is not zero.

**Remark 4.3.** In terms of representation theory, Property (#) translates as

$$\langle \chi^1_{can}, \chi\rangle \neq 0 \qquad \text{and} \qquad \langle \chi^2_{can}, \overline{\chi}\rangle \neq 0 \implies \deg(\chi) = 1$$

for each irreducible character $\chi$, where $\chi^i_{can}$ is the character of the canonical representation of $C_i$, $i = 1, 2$.
Thus, once $\chi^1_{can}$ and $\chi^2_{can}$ are determined using the Chevalley-Weil formula, verifying whether Property (#) holds reduces to a simple numerical computation.

Let us suppose now Property (#) is satisfied, namely each irreducible character $\chi$ of $G$ for which (4.2) is not zero has degree one.
In Subsection 4.3 we explain how to compute the self-intersection of the mobile part $M$ of $|K_{C_1\times C_2}|^G$.
Note that the difference $M^2 - \widehat{M}^2$ is the sum of the correction terms arising from each isolated base-point of $M$.

To finish the computation of the degree, whenever $p_g(S) = 3$, we use iteratively for each base point of $M$ the Correction Term formula 4.28 in Subsection 4.5, which provides the correction term of each base point to the difference $M^2 - \widehat{M}^2$. Such formula is a generalization of the formula presented in [20] and it seems of independent interest, so that it is presented in a more general setting.
Once we have determined both $M^2$ and $M^2 - \widehat{M}^2$, then the degree of the canonical map of $S$ is obtained by rearranging formula (4.1) as follows

$$\deg(\Phi_S) = \frac{1}{|G|} \cdot \left( M^2 - \left( M^2 - \widehat{M}^2 \right) \right).$$

4.1. **Base locus of isotypic components of canonical representations of actions on curves.** Let $C$ be a Riemann surface, $G < \mathrm{Aut}(C)$ be a finite group, $C' := C/G$ its quotient, and let $\lambda \colon C \to C'$ be the quotient map.
$G$ acts on $H^{1,0}(C)$ via the canonical representation:

$$(g \cdot \omega)_p := (dg^{-1})_p \omega_{g^{-1}\cdot p},$$

Let us denote by $\chi_{can}$ the character of the canonical representation, which takes the name of *canonical character*. The canonical representation can be splitted as a direct sum of irreducible representations:

$$H^{1,0}(C) = \bigoplus_{\chi \in Irr(G)} H^{1,0}(C)^{\chi}.$$

Here $H^{1,0}(C)^{\chi}$ is the *isotypic component* of $H^{1,0}(C)$ of character $\chi$, namely it is a $G$-invariant subspace such that the restriction of the canonical representation is isomorphic to $\langle \chi_{can}, \chi \rangle$-times the irreducible representation given by the character $\chi$.
In terms of characters, the above splitting translates as

$$\chi_{can} = \sum_{\chi \in Irr(G)} \langle \chi_{can}, \chi \rangle \cdot \chi.$$

We shall use the algorithm developed in [24] and implemented in the computational algebra system MAGMA to compute the canonical character $\chi_{can}$ of any Galois branched covering.

The aim of this section is to investigate the base locus of the associated subsystem $|K_C|^{\chi}$ given by the isotypic component $H^{1,0}(C)^{\chi}$. Let us give first some preliminary results.

**Notation:** Given a point $q \in C'$, the divisor $\lambda^{-1}(q)$ is considered with the reduced structure.

**Lemma 4.4.** *Consider a $G$-invariant subspace $W \subseteq H^{1,0}(C)$. For any $p \in \lambda^{-1}(q)$, let $t_p$ be the minimal order of vanishing of a 1-form in $|W|$ at $p$. Then all $t_p$ are equal to the same number, denoted by $t_q$. Therefore the base locus of $|W|$ is a union of orbits*

$$Bs(|W|) = \sum_q t_q \lambda^{-1}(q).$$

*Furthermore, there exists a general form $\omega \in W$ vanishing of order exactly $t_q$ at each $p \in \lambda^{-1}(q)$.*

*Proof.* For every point $p \in \lambda^{-1}(q)$, it there exists a 1-form $\omega_p$ in $W$ vanishing at $p$ with order $t_p$, by the definition of $t_p$. Given $g \in G$, then $g \cdot \omega_p$ belongs to the invariant subspace $W$ too, and it vanishes at $g \cdot p$ with multiplicity $t_p$, so that $t_{g \cdot p} \leq t_p$. Hence all $t_p$ are equal to the same number, denoted as $t_q$.

We observe that a generic linear combination $\omega$ of the obtained $|\lambda^{-1}(q)|$ 1-forms $\omega_p$ vanishes with order $t_q$ at each point of $\lambda^{-1}(q)$. □

**Remark 4.5.** Let $\omega \in W$ be a 1-form of the Lemma 4.4, with vanishing order $t_q$ at each point $p \in \lambda^{-1}(q)$. Given $g \in G$, then $g \cdot \omega \in W$ is a 1-form with vanishing order $t_q$ at each point $p \in \lambda^{-1}(q)$.

Let $H^{1,0}(C)^\chi$ be the isotypic component of $H^{1,0}(C)$ of irreducible character $\chi$.

**Lemma 4.6.** *Let $f \in \mathcal{M}(C/G) = \mathcal{M}(C)^G$ be a non-zero invariant meromorphic function. Denote by $H^{1,0}(C)^\chi_f$ the subspace of $H^{1,0}(C)^\chi$ consisting of forms $\omega$ such that $f\omega$ is a holomorphic form. Then*

$$f\colon H^{1,0}(C)^\chi_f \to f \cdot H^{1,0}(C)^\chi_f \subseteq H^{1,0}(C), \qquad \omega \mapsto f\omega \tag{4.3}$$

*is a $G$-equivariant isomorphism. In particular, $f \cdot H^{1,0}(C)^\chi_f$ is a $G$-invariant subspace of $H^{1,0}(C)^\chi$.*

*Proof.* $H^{1,0}(C)^\chi_f$ is $G$-invariant: given $g \in G$ and $\omega \in H^{1,0}(C)^\chi_f$, then $f(g \cdot \omega) = g \cdot (f\omega)$ is holomorphic since $f$ is $G$-invariant, and $f\omega$ is holomorphic. This shows immediately also that the map of (4.3) is $G$-equivariant. From Schur Lemma, then the image of (4.3) is contained in $H^{1,0}(C)^\chi$. However, $f$ is not the zero function, so (4.3) is injective. □

**Definition 4.7.** Let $X$ be a Riemann surface and $q \in X$. Let us define

$$k_q := \min\left\{m \in \mathbb{N} : h^0(X, mq) \geq 2\right\}$$

be the *minimal non-gap of $q$*. $k_q$ is therefore the smallest number such that $X$ admits a non-constant meromorphic function $f$ with only one pole at $q$, of order $-k_q$.

**Remark 4.8.** From Riemann-Roch theorem we have

$$h^0(X, (g(X)+1)q) = h^0(X, K - (g(X)+1)q) + 2 \geq 2.$$

Therefore

$$k_q \leq g(X) + 1.$$

In other words, $k_q$ is the minimum of the complement of the set of the Weierstrass gaps for $q$. In particular, $k_q = g(X) + 1$, if $q$ is not a Weierstrass point, or $k_q < g(X) + 1$, otherwise.

Let $q \in C'$ be a branch point of $\lambda$. The stabilizers of the points lying on $q$ are cyclic subgroups of $G$ and they are conjugated to each other. Thus the order of

the stabilizers depends only on $q$, denoted as $m_q$.
We remind the definition

**Definition 4.9.** Let us fix a point $p \in \lambda^{-1}(q)$. Given a generator $h$ of $Stab(p)$, there exists a coordinate $z$ in $C$ such that the action of $h$ in a neighborhood of $p$ corresponds to $z \to \lambda z$, where $\lambda$ is one of the $m_q$-roots of the unity. This gives a bijection among the primitive $m_q$-roots of the unity and the generators of $Stab(p)$. We denote by *local monodromy* of $p$ the unique generator of $Stab(p)$ acting by $z \to e^{\frac{2\pi i}{m_q}} z$.

**Remark 4.10.** The *local monodromy* of another point $g \cdot p$ over $q$ is the conjugate $ghg^{-1}$ of $h$. In other words, the *local monodromy* of points lying over $q$ are conjugated to each other.

Lemma 4.4 applies to $H^{1,0}(C)^\chi$, so the base locus of $|K_C|^\chi$ is

$$Bs(|K_C|^\chi) = \sum_q t_q^\chi \lambda^{-1}(q),$$

for some natural integers $t_q^\chi$, that we still need to determine.

We denote by $\rho_\chi$ an irreducible representation of $G$ of character $\chi$.
We have the following

**Lemma 4.11.** *Let us fix a point $q \in C/G$ of ramification index $m_q$. Let $h$ be the local monodromy of a point $p \in \lambda^{-1}(q)$, hence $o(h) = m_q$. It there exists*

$$a_q^\chi \in \{j \in [0, \dots, m_q - 1] \colon e^{\frac{2\pi i}{m_q} j} \in Spec(\rho_\chi(h))\}$$

*and a non-negative integer $0 \leq k_q^\chi < k_q \leq g(C/G) + 1$ such that*

$$t_q^\chi = m_q - a_q^\chi - 1 + k_q^\chi m_q,$$

*where $k_q$ is the minimal non-gap of $q$ in the Definition 4.7.*
*The values $a_q^\chi$ and $k_q^\chi$ depends only from $q$ and $\chi$ and not by the choice of $p \in \lambda^{-1}(q)$.*

*Proof.* We observe that the action on $H^{1,0}(C)^\chi$ of $h$ is diagonalizable, and its spectrum is contained in the set of the $m_q$-roots of the unity. Hence the action of $h$ decomposes $H^{1,0}(C)^\chi$ as

$$H^{1,0}(C)^\chi = \bigoplus_{j=0}^{m_q-1} V_j,$$

where $V_j$ is the eigenspace of eigenvalue $\xi^j$, and $\xi$ is the first $m_q$-root of the unity ($V_j$ may be zero, whenever $\xi^j$ is not an eigenvalue of $h$).

Let $\omega_j \in V_j$ be an eigenvector. We determine the vanishing order of $\omega_j$ at the point $p$. By definition of local monodromy, it there exists a local coordinate $z$ such that the action of $h$ in a neighborhood of $p$ is $z \mapsto \xi z$. We write $\omega_j = f(z)dz$

locally around this neighborhood of $p$. We get

$$\begin{aligned}\xi^j f(z)dz &= h\cdot(f(z)dz)\\ &= (h^{-1})^*(f(z)dz)\\ &= f(\xi^{m_q-1}z)\xi^{m_q-1}dz.\end{aligned}$$

Hence $f$ satisfies $f(\xi^{m_q-1}z)=\xi^{j+1}f(z)$, forcing it to be $f=z^{m_q-j-1}g(z^{m_q})$, for some holomorphic function $g$. Hence $\mathrm{ord}_p(\omega_j)$ is congruent to $m_q-j-1$ modulo $m_q$.

Applying Lemma 4.4 to $W=H^{1,0}(C)^\chi$ we find a form $\omega\in H^{1,0}(C)^\chi$ with vanishing order $t_q^\chi$ at each point of $\lambda^{-1}(q)$. Let us write $\omega$ as a $\omega=\sum_{j=0}^{m_q-1}\omega_j$, with $\omega_j\in V_j$. Since each $\omega_j$ has different order at $p$, then

$$t_q^\chi=\mathrm{ord}_p(\omega)=\min_{\omega_j\neq 0}\{\mathrm{ord}_p(\omega_j)\}.$$

In other words, it there exists $j_0\in[0,\dots,m_q-1]$ such that $t_q^\chi=\mathrm{ord}_p(\omega_{j_0})$.

Since $\omega_{j_0}$ is an eigenvector of eigenvalue $\xi^{j_0}$, then $t_q^\chi=\mathrm{ord}_p(\omega_{j_0})$ is congruent to $m_q-j_0-1$ modulo $m_q$; let us say $t_q^\chi=m_q-j_0-1+k_{j_0}m_q$, for some non-negative integer $k_{j_0}$.

We claim that $k_{j_0}<k_q$. By contradiction, if $k_{j_0}\geq k_q$, then we use the definition of $k_q$ to pick up a meromorphic function $f\in\mathcal{M}(C/G)=\mathcal{M}(C)^G$ with only one pole at $q$ of order $\mathrm{ord}_q(f)=-k_q$. In this case, then $f\omega$ is a holomorphic form. Indeed, by definition of $f$, the only poles of $f\omega$ that may occur lie on $\lambda^{-1}(q)$, but the order of $f\omega$ at each $g\cdot p\in\lambda^{-1}(q)$ is

$$\begin{aligned}\mathrm{ord}_{g\cdot p}(f\omega) &= \mathrm{ord}_{g\cdot p}(\omega)+\mathrm{ord}_{g\cdot p}(f)\\ &= t_q^\chi-k_qm_q\\ &= m_q-j_0-1+(k_{j_0}-k_q)m_q\geq 0.\end{aligned}$$

Furthermore, from Lemma 4.6, then $f\omega\in H^{1,0}(C)^\chi$. However, this would contradict the definition of $t_q^\chi$, since $\mathrm{ord}_p(f\omega)=t_q^\chi-k_qm_q<t_q^\chi$.

To summarize, we have proved

$$t_q^\chi=m_q-j_0-1+k_{j_0}m_q,$$

where $j_0$ is one of the integers such that $\xi^{j_0}\in Spec(\rho_\chi(h))$, and $k_{j_0}<k_q$.

It is straightforward to see that such integers $j_0$ and $k_{j_0}$ do not depend from the choice of $p\in\lambda^{-1}(q)$. □

**Theorem 4.12.** *(Base locus formula) The base locus of $|K_C|^\chi$ is*

$$Bs(|K_C|^\chi)=\sum_q\left(m_q-a_q^\chi-1+k_q^\chi m_q\right)\lambda^{-1}(q),$$

*where the non-negative integers $a_q^\chi$ and $k_q^\chi$ are those defined in Lemma 4.11.*

*Proof.* It suffices to apply Lemma 4.11 to each point $q\in C/G$. □

**Remark 4.13.** Under suitable assumptions it is possible to determine exactly $a_q^\chi$ and $k_q^\chi$.

For instance, if $C/G \cong \mathbb{P}^1$, then $k_q = g(C/G)+1 = 1$, for any $q \in \mathbb{P}^1$. Hence $k_q^\chi = 0$, and we get

$$t_q^\chi = m_q - a_q^\chi - 1.$$

Moreover, if one of the following holds

- $\chi$ is an irreducible character of degree 1, or
- the local monodromy $h$ is in the centre of $G$,

then $\rho_\chi(h) = \frac{\chi(h)}{\chi(1)} \cdot \mathrm{Id}$ is a multiple of the identity.

This is obvious when the character has degree one. Instead, when the local monodromy is central, this is a result we take from [15].

Under one of these two conditions, then $a_q^\chi \in [0, \dots, m_q - 1]$ is the only integer such that $\chi(h) = e^{\frac{2\pi i}{m_q} a_q^\chi} \chi(1)$.

We deduce then the following immediate consequence from Theorem 4.12 and Remark 4.13:

**Corollary 4.14.** *Assume $C/G \cong \mathbb{P}^1$, and $\chi$ is an irreducible character of degree 1. Then*

$$Bs(|K_C|^\chi) = \sum_q \left(m_q - a_q^\chi - 1\right) \lambda^{-1}(q),$$

*where $a_q^\chi \in [0, \dots m_q - 1]$ is the only non-negative integer such that $\chi(h) = e^{\frac{2\pi i}{m_q} a_q^\chi}$, with $h$ local monodromy of a point $p$ over $q$.*

4.2. **The canonical system of a product-quotient surface.** Let us consider a product-quotient surface $S$ given by a pair of curves $C_1$ and $C_2$ and a finite group $G$ acting (faithfully) on both of them. Let $X := (C_1 \times C_2)/G$ be the the quotient model of $S$.

According to the previous section, then $G$ induces the canonical representation on $H^{1,0}(C_i)$; let $\chi_{can}^i$ be their canonical characters respectively, $i = 1, 2$.

**Theorem 4.15.** *Every $G$-invariant global holomorphic 2-form of $C_1 \times C_2$ extends uniquely to a global holomorphic 2-form on the minimal resolution of the singularities $\rho\colon S \to X$ of $X$. It holds*

$$H^{2,0}(S) = H^{2,0}(C_1 \times C_2)^G = \bigoplus_{\chi \in Irr(G)} \left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G. \tag{4.4}$$

*Furthermore,*

$$p_g(S) = \sum_{\chi \in Irr(G)} \langle \chi_{can}^1, \chi \rangle \cdot \langle \chi_{can}^2, \overline{\chi} \rangle.$$

*Proof.* Denote by $X^\circ$ the smooth locus of $X$, i.e. the locus of the image of that points of $C_1 \times C_2$ with trivial stabilizer. Each global holomorphic 2-form of $X^\circ$ extends uniquely to a global holomorphic 2-form of $C_1 \times C_2$, via the pullback map $\lambda_{12}^*\colon H^{2,0}(X^\circ) \to H^{2,0}(C_1 \times C_2)$, resulting a monomorphism onto the invariant subspace $H^{2,0}(C_1 \times C_2)^G$. On the other side, the minimal resolution of the singularities $\rho\colon S \to X$ is an isomorphism on $X^\circ$, hence $\left(\rho^{-1}\right)^*\colon H^{2,0}(S) \to H^{2,0}(X^\circ)$ is a monomorphism. Furthermore, each global holomorphic 2-form on the smooth

locus $X^\circ$ of $X$ extends uniquely to a global holomorphic 2-form on $S$, by Freitag's theorem [21, Satz 1], so $\left(\rho^{-1}\right)^*$ is an epimorphism too.
Thus $H^{2,0}(S)$ is sent isomorphically via $\lambda_{12}^* \circ (\rho^{-1})^*$ onto the invariant subspace $H^{2,0}(C_1 \times C_2)^G \subseteq H^{2,0}(C_1 \times C_2)$. Finally, by applying Künneth formula and writing $H^{1,0}(C_i)$ as the direct sum of isotypic components, we get

$$H^{2,0}(C_1 \times C_2)^G = \bigoplus_{\chi,\eta \in Irr(G)} \left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^\eta\right)^G.$$

Formula (4.4) follows just from Schur lemma. Indeed, the dimension of any piece of the sum is $\langle \chi^1_{can}, \chi\rangle \cdot \langle \chi^2_{can}, \eta\rangle \cdot \langle \chi\eta, 1\rangle$. However $\langle \chi\eta, 1\rangle = \langle \chi, \overline{\eta}\rangle$, which is equal to 1 only for $\eta = \overline{\chi}$, and 0 otherwise. □

**Remark 4.16.** Using an analogous proof such as that of Theorem 4.15 one can say in general that

$$H^{i,0}(S) = H^{i,0}(C_1 \times C_2)^G$$

by Freitag's theorem [21, Satz 1]. Hence, another immediate consequence firstly observed by Serrano in [33, Prop. 2.2] is a formula for the irregularity of $S$:

$$q(S) = g(C_1/G) + g(C_2/G).$$

In particular, $S$ is regular if and only if $C_i/G \cong \mathbb{P}^1$.

Let us remind the following classical lemma of representation theory:

**Lemma 4.17.** *Let us consider an irreducible representation $\phi_\chi$ afforded by a character $\chi$, of degree $n := \chi(1)$. Consider a basis $v_1, \dots, v_n$ of $V$ and its dual basis $e_1, \dots, e_n$ of $V^*$. Then $(V \otimes V^*)^G$ is one-dimensional and it is generated by $v_1 \otimes e_1 + \dots + v_n \otimes e_n$.*

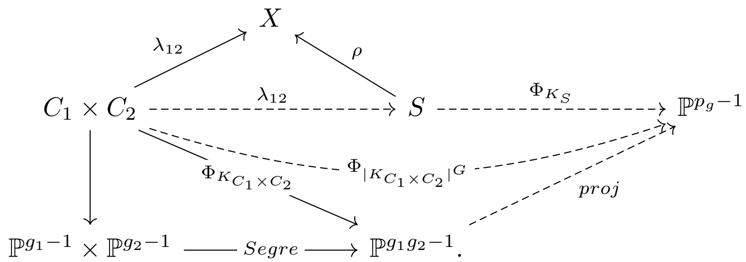
We use the previous lemma to describe a basis of $(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}})^G$.

**Remark 4.18.** Let us consider an irreducible representation $\phi_\chi \colon G \to GL(V)$ of character $\chi$. Let $n := \chi(1)$ be the degree of $\phi_\chi$. Then $H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$ is the direct sum of certain number of copies of $V \otimes V^*$ (the exact number of copies is $\langle \chi^1_{can}, \chi\rangle \cdot \langle \chi^2_{can}, \overline{\chi}\rangle$). Consequently its invariant subspace $(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}})^G$ is a direct sum of the same number of copies of the invariant subspace $(V \otimes V^*)^G$. Let us fix a basis $\{\omega_1, \dots, \omega_n\}$ of $V$ and the (dual) basis $\{\eta_1, \dots, \eta_n\}$ on $V^*$. Hence, denote by $\{\omega_1^k, \dots, \omega_n^k\}$ the corresponding basis of the $k$-th copy of $V$ on $H^{1,0}(C_1)^\chi$, $k = 1, \dots, \langle \chi^1_{can}, \chi\rangle$ [resp. by $\{\eta_1^l, \dots, \eta_n^l\}$ the corresponding basis of the $l$-th copy of $V^*$ on $H^{1,0}(C_2)^{\overline{\chi}}$, $l = 1, \dots, \langle \chi^2_{can}, \overline{\chi}\rangle$]. Lemma 4.17 applies for any copy of $(V \otimes V^*)^G$, so that

$$(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}})^G = \bigoplus_{k,l} \langle \omega_1^k \otimes \eta_1^l + \dots + \omega_n^k \otimes \eta_n^l \rangle. \tag{4.5}$$

**Definition 4.19.** We denote by $|K_{C_1 \times C_2}|^G$ the linear subsystem of the canonical system of $C_1 \times C_2$ given by the subspace of invariant 2-forms of $C_1 \times C_2$.

We give a theoretical description of the canonical map $\Phi_{K_S}$ of $S$. From Theorem 4.15, the (rational) map $\Phi_{K_S} \circ \lambda_{12}$ is induced by the linear subsystem $|K_{C_1 \times C_2}|^G$. The situation is the following:




Let us fix a basis of $H^{1,0}(C_1)$ and $H^{1,0}(C_2)$. Then $\Phi_{K_S} \circ \lambda_{12}$ is the composition of the product of the canonical maps of $C_1$ and $C_2$ with the Segre embedding in $\mathbb{P}^{g_1 g_2 - 1}$, together with the projection map *proj*. This latter map sends a basis of 2-forms of $C_1 \times C_2$ to a basis of invariant 2-forms defining $\Phi_{K_S}$.
We can use Remark 4.18 to give an explicit description of *proj*, which is defined in coordinates as follows:
Let us fix coordinates ${}^{\chi}x_{ij}^{kl}$ on $\mathbb{P}^{g_1 g_2 - 1}$, with $1 \leq i, j \leq \chi(1)$, and $1 \leq k \leq \langle \chi^1_{can}, \chi \rangle$, $1 \leq l \leq \langle \chi^2_{can}, \overline{\chi} \rangle$. Then

$$proj\left(\left({}^{\chi}x_{ij}^{kl} : \chi, i, j, k, l\right)\right) = \left({}^{\chi}x_{11}^{kl} + \cdots + {}^{\chi}x_{nn}^{kl} : \quad \chi \in Irr(G), n = \chi(1), k, l\right).$$

4.3. **Base locus of the invariant subsystem $|K_{C_1 \times C_2}|^G$.** Given an irreducible character $\chi \in Irr(G)$, we have the following series of inclusions

$$\left(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G \subseteq H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}} \subseteq H^{2,0}(C_1 \times C_2).$$

Let us define the associated subsystems of $|K_{C_1 \times C_2}|$ given by these subspaces.

**Definition 4.20.** We denote by $|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}$ and by $\left(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}\right)^G$ the associated subsystems of the canonical linear system of $C_1 \times C_2$ given by $H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}$ and $\left(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G$ respectively.

Theorem 4.15 permits us to describe the base locus of $|K_{C_1 \times C_2}|^G$ in terms of the base locus of its pieces $\left(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}\right)^G$, $\chi \in Irr(G)$. Precisely, we have

$$Bs(|K_{C_1 \times C_2}|^G) = \bigcap_{\langle \chi^1_{can}, \chi \rangle \neq 0, \ \langle \chi^2_{can}, \overline{\chi} \rangle \neq 0} Bs\left(\left(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}\right)^G\right). \tag{4.6}$$

**Notation:** Let us denote by

$$B_q^{vert} := \{q\} \times C_2/G, \qquad \text{and} \qquad B_l^{hor} := C_1/G \times \{l\},$$

where $q \in C_1/G$ and $l \in C_2/G$. Instead, $R_q^{vert}$ and $R_l^{hor}$ denote the reduced inverse images on $C_1 \times C_2$ of $B_q^{vert}$ and $B_l^{hor}$:

$$R_q^{vert} := \frac{1}{m_q}(\lambda \circ \lambda_{12})^*(\{q\} \times C_2/G), \qquad R_l^{hor} := \frac{1}{m_l}(\lambda \circ \lambda_{12})^*(C_1/G \times \{l\}).$$

**Remark 4.21.** With this notation, then the branch locus of $\lambda \circ \lambda_{12} \colon C_1 \times C_2 \to C_1/G \times C_2/G$ is the grid

$$B_q^{vert} := \{q\} \times C_2/G, \qquad \text{and} \qquad B_l^{hor} := C_1/G \times \{l\}$$

with $q \in Crit(\lambda_1)$ and $l \in Crit(\lambda_2)$.

*Base Locus formula* theorem 4.12 provides a formula for the base locus of $|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}$.

**Theorem 4.22.** *The (schematic) base locus of the linear subsystem $|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}$ of $|K_{C_1 \times C_2}|$ is pure in codimension 1 and is equal to*

$$Bs(|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}) = \sum_{q \in Crit(\lambda_1)} t_q^\chi R_q^{vert} + \sum_{l \in Crit(\lambda_2)} t_l^{\overline{\chi}} R_l^{hor} \tag{4.7}$$

*where $t_q^\chi$ and $t_l^{\overline{\chi}}$ are the non-negative integers of Lemma 4.11.*

**Corollary 4.23.** *Let $\chi$ be a character of degree 1. Then*

$$(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}})^G = H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$$

*and the base locus of its associated linear subsystem $\left(|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}\right)^G = |K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}$ is given by the formula (4.7) of Theorem 4.22.*

*Assume furthermore that $C_i/G \cong \mathbb{P}^1$, for $i = 1, 2$. Then $t_q^\chi$ and $t_l^{\overline{\chi}}$ of (4.7) are the unique non-negative integers with $0 \le t_q^\chi \le m_q - 1$ and $0 \le t_l^{\overline{\chi}} \le m_l - 1$ satisfying*

$$\chi(h) = e^{\frac{2\pi i}{m_q}(m_q - t_q^\chi - 1)} \qquad \text{and} \qquad \chi(g) = e^{\frac{2\pi i}{m_l}(t_l^{\overline{\chi}} + 1)},$$

*where $h$ is the local monodromy of a point over $q$, and $g$ is the local monodromy of a point over $l$.*

*Proof.* The first sentence is straightforward, since every $v \otimes w \in H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$ is $G$-invariant

$$g \cdot (v \otimes w) = (\chi(g)v) \otimes (\overline{\chi}(g)w) = |\chi(g)| v \otimes w = v \otimes w.$$

The rest of the thesis follows from Remark 4.13. ∎

**Lemma 4.24.** *Suppose $S$ satisfies Property (#). Then the fixed part of the linear system $|K_{C_1 \times C_2}|^G$ is*

$$\begin{aligned} Fix\left(|K_{C_1 \times C_2}|^G\right) = &\sum_{q \in Crit(\lambda_1)} \left( \min_{\chi \colon \langle \chi_{can}^1, \chi \rangle \neq 0, \langle \chi_{can}^2, \overline{\chi} \rangle \neq 0} t_q^\chi \right) R_q^{vert} + \\ &\sum_{l \in Crit(\lambda_2)} \left( \min_{\chi \colon \langle \chi_{can}^1, \chi \rangle \neq 0, \langle \chi_{can}^2, \overline{\chi} \rangle \neq 0} t_l^{\overline{\chi}} \right) R_l^{hor}. \end{aligned} \tag{4.8}$$

*Proof.* The fixed part of $|K_{C_1\times C_2}|^G$ is the common divisor of the fixed parts of those pieces $\left(|K_{C_1}|^{\chi}\otimes|K_{C_2}|^{\overline{\chi}}\right)^G$ that are non-empty, for $\chi$ irreducible character. By Property (#) , then $\chi$ is of degree 1, hence Corollary 4.23 applies and the fixed part of $\left(|K_{C_1}|^{\chi}\otimes|K_{C_2}|^{\overline{\chi}}\right)^G$ is amount to

$$\sum_{q\in Crit(\lambda_1)} t_q^{\chi} R_q^{vert} + \sum_{l\in Crit(\lambda_2)} t_l^{\overline{\chi}} R_l^{hor}.$$

The common divisor of these fixed parts is the right member of (4.8). □

Let $M$ be the mobile part of $|K_{C_1\times C_2}|^G$. By definition of $M$, then

$$M \equiv K_{C_1\times C_2} - Fix(|K_{C_1\times C_2}|^G).$$

Suppose $S$ satisfies Property (#). Thus $Fix(|K_{C_1\times C_2}|^G)$ is a union of fibres from Lemma 4.8. To compute $M^2$ is then sufficient to know the intersection product of

$$K_{C_1\times C_2}\cdot R_q^{vert},\qquad K_{C_1\times C_2}\cdot R_l^{hor},\qquad \left(R_q^{vert}\right)^2,\qquad \left(R_l^{hor}\right)^2,\qquad R_q^{vert}\cdot R_l^{hor}.$$

We compute them.
$R_q^{vert}$ can be written as sum of $|G|/m_q$ components $\{g\cdot p\}\times C_2$, with $p$ point over $q$, and $g\in G$. $\{g\cdot p\}\times C_2$ has self-intersection zero (since two points are always homologous on a connected variety, and then the fibres of $C_1\times C_2\to C_1$ are always numerically equivalent). Thus we can use *genus formula* to get

$$K_{C_1\times C_2}\cdot(\{g\cdot p\}\times C_2) = 2g(C_2)-2-(\{g\cdot p\}\times C_2)^2 = 2g(C_2)-2.$$

The same reasoning works for an horizontal divisor $R_l^{hor}$. Thus, we have got

$$K_{C_1\times C_2}\cdot R_q^{vert} = \frac{|G|}{m_q}\left(2g(C_2)-2\right),\qquad K_{C_1\times C_2}\cdot R_l^{hor} = \frac{|G|}{m_l}\left(2g(C_1)-2\right).$$

Analogously,

$$\left(R_q^{vert}\right)^2 = \left(R_l^{hor}\right)^2 = 0,\qquad \text{and}\qquad R_q^{vert}\cdot R_l^{hor} = \frac{|G|^2}{m_q m_l}.$$

4.4. **A formula for the degree of the canonical map.** In the previous subsection we have seen that the (a priori rational) map $\Phi_{K_S}\circ\lambda_{12}$ is induced by the linear subsystem $|K_{C_1\times C_2}|^G$, which is generated by $p_g$ invariant 2-forms defining $\Phi_{K_S}$:

$$\begin{array}{ccccc} C_1\times C_2 & \overset{\lambda_{12}}{\dashrightarrow} & S & \overset{\Phi_S}{\dashrightarrow} & \mathbb{P}^{p_g-1} \end{array}\qquad \Phi_{|K_{C_1\times C_2}|^G}: C_1\times C_2 \dashrightarrow \mathbb{P}^{p_g-1}$$

We *resolve the indeterminacy* of $\Phi_{|K_{C_1\times C_2}|^G} = \Phi_{K_S}\circ\lambda_{12}$ by a sequence of blowups:

$$\begin{array}{ccc} \widehat{C_1\times C_2} & \longrightarrow & C_1\times C_2 \\ & \searrow^{\Phi_{\widehat{M}}} & \Big\downarrow{\scriptstyle \Phi_{|K_{C_1\times C_2}|^G}} \\ & & \mathbb{P}^{p_g-1}. \end{array}$$

Here the morphism $\Phi_{\widehat{M}}$ is induced by the base-point free linear system $\widehat{M}$ obtained as follow: let $M$ be the mobile part of $|K_{C_1\times C_2}|^G$.
We blow up the base-points of $M$, take the pullback of $M$ and remove the fixed part of this new linear system. We repeat the procedure, until we obtain a base-point free linear system $\widehat{M}$.

**Lemma 4.25.** *The map $\Phi_{K_S}$ is not composed with a pencil if and only if $\widehat{M}^2$ is positive.*

*Proof.* The map $\Phi_{K_S}$ is composed with a pencil if and only if $\Phi_{\widehat{M}}$ is composed with a pencil. The image $\Sigma$ of $\Phi_{\widehat{M}}$ is a curve if and only if we are able to pick-up two general hyperplanes $H_1$ and $H_2$ of $\mathbb{P}^{p_g-1}$ such that $H^2_{|\Sigma} = H_1 \cdot H_2 \cdot \Sigma = 0$. However, $\widehat{M} = \Phi^*_{\widehat{M}}(H)$, hence $H^2_{|\Sigma}$ is zero if and only if $\widehat{M}^2$ is equal to zero. □

Let us suppose $\widehat{M}^2 > 0$, so that $\Phi_{K_S}$ has image $\Sigma$ of dimension 2. In this case, then $\Phi_{\widehat{M}}$ is a finite morphism, and by projection formula

$$\widehat{M}^2 = \deg(\Phi_{\widehat{M}})\deg(\Sigma) = \deg(\Phi_{K_S})\deg(\Sigma)|G|,$$

which gives Formula (4.1).

4.5. **The correction term to the self-intersection of a 2-dimensional linear system with only isolated base points.** As remarked in the introduction of this chapter, $M^2 - \widehat{M}^2$ is the sum of the correction terms arising from each isolated base-point of $M$, the mobile part of the linear subsystem $|K_{C_1\times C_2}|^G$.

The contribution to the correction term of any isolated base-point may be easily computed whenever $S$ satisfies Property (#).

Let us fix a base-point $(p_1, p_2) \in C_1 \times C_2$ of the mobile part $M$. The point $p_1$ is over $q \in C_1/G$ and $p_2$ is over $l \in C_2/G$. Let us fix an irreducible character $\chi$. We can always choose a general basis of $H^{1,0}(C_1)^\chi$ such that each one-form of the basis has the minimum vanishing order $t^\chi_q$ at $p_1$, which is the natural integer computed in Lemma 4.11.
Similarly, we can choose a general basis of $H^{1,0}(C_2)^{\overline{\chi}}$ such that each one-form of the basis has minimum vanishing order $t^{\overline{\chi}}_l$ at $p_2$. The choice of this pair of bases gives via tensor product a natural basis of $H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$, which is a $G$-invariant subspace since Property (#) holds, namely $\chi$ is of degree one. This permits us to conclude that the divisors spanning the linear subsystem $|K_{C_1\times C_2}|^G$ can be written in a neighbourhood of $(p_1, p_2)$ as

$$t^\chi_q R^{vert}_q + t^{\overline{\chi}}_l R^{hor}_l, \qquad \chi \text{ such that } \qquad \langle \chi^1_{can}, \chi\rangle \neq 0, \langle \chi^2_{can}, \overline{\chi}\rangle \neq 0.$$

Finally, it is sufficient to remove the fixed part of $|K_{C_1\times C_2}|^G$ computed in Lemma 4.24 to get how the divisors spanning $M$ are written in a neighbourhood of $(p_1, p_2)$. So, the linear system $M$ is spanned by $p_g$ divisors locally near $(p_1, p_2)$ of the form

$$a_1 R^{vert}_q + b_1 R^{hor}_l, \qquad \dots \qquad a_{p_g} R^{vert}_q + b_{p_g} R^{hor}_l.$$

Since we assumed that $(p_1, p_2)$ is a base-point and $M$ has not fixed components, then without loss of generality $a_1 = b_2 = 0$.
Note that $R_q^{vert}$ and $R_l^{hor}$ are smooth and intersect transversally at $(p_1, p_2)$.

In Theorem 4.28 we give a general formula to compute directly the contribution of $(p_1, p_2)$ to the correction term $M^2 - \widehat{M}^2$ whenever $p_g$ is equal to three.

The rest of this subsection proves Theorem 4.28.

Let us consider a slightly more general setting: let $M$ be a (not necessarily complete) two-dimensional linear system on a surface $S$ spanned by $D_1$, $D_2$, and $D_3$. Assume that $M$ has only isolated base-points, smooth for $S$, and that in a neighborhood of a basepoint $p$ we can write the divisors $D_i$ as

$$D_1 = aH, \quad D_2 = bK \quad \text{and} \quad D_3 = cH + dK.$$

Here $H$ and $K$ are reduced, smooth, and intersect transversally at $p$ and $a, b, c, d$ are non-negative integers, $b \leq a$.

Let $\widehat{M}$ be the linear system obtained as follows: we blow-up the basepoint $p$, take the pullback of the mobile part of $M$ and remove the fixed part of this new linear system. If an infinitely near point of $p$ is a base-point for this linear system, then repeat the procedure, until we obtain a (not necessarily complete) linear system $\widehat{M}$ such that no infinitely near point of $p$ is a base point of $\widehat{M}$.

**Definition.** The linear system $\widehat{M}$ is called *strict transform* of $M$ at $p$.

Firstly, we present a stronger version of [19, Lemma 2].

**Lemma 4.26.** *Assume that $bc + ad \geq ab$. Then $M^2 - \widehat{M}^2 = ab$.*

*Proof.* We prove the lemma by induction on $(a, b)$, with $b \leq a$. Here we are considering the lexicographic order $\leq$ defined on the lower half plane $\Delta^{\geq} := \{(a, b) \colon a \geq b\} \subseteq \mathbb{N} \times \mathbb{N}$ as follows:

$$(a', b') \leq (a, b) \text{ if and only if } a' < a \text{ or } a' = a \text{ and } b' \leq b.$$

In this case, $\Delta^{\geq}$ admits the *well-ordering principle* and so it holds the *mathematical induction*.
Suppose that $(a, b) = 0$. Then $M$ is base-point free and so $\widehat{M}^2 = M^2 = M^2 - ab$. Now suppose that the statement is true for $(a', b') < (a, b)$. We aim to prove it for $(a, b)$. We blow up the base-point $p$, take the pullback of the divisors $D_i$, and remove the fixed part, which is the exceptional divisor $bE$ of the blowup. In fact the pullback of $D_3$ contains $c + d$ times $E$ and $c + d \geq b$, thanks to $b \leq a$ and to the assumption $bc + ad \geq ab$:

$$a(c + d) \geq bc + ad \geq ab, \qquad \text{so} \qquad c + d \geq b.$$

Restricted to the preimage of our neighborhood of $p$, these divisors are:

$$a\widehat{H} + (a - b)E, \qquad b\widehat{K} \qquad \text{and} \qquad c\widehat{H} + d\widehat{K} + (c + d - b)E.$$

Here, $\widehat{H}$ and $\widehat{K}$ are the strict transforms of $H$ and $K$. Let $\widehat{M}$ be the linear system generated by these three divisors, then $\widehat{M}^2 = M^2 - b^2$. If $a = b$ or $b = 0$, then $\widehat{M}$ is base-point free and we are done. Otherwise, on the preimage, the linear system

$\widehat{M}$ has precisely one new base-point: the intersection point of $\widehat{K}$ and $E$. Locally near this point the three divisors spanning $\widehat{M}$ are:

$$(a-b)E, \qquad b\hat{K} \qquad \text{and} \qquad d\hat{K}+(c+d-b)E.$$

We need to distinguish two cases, when $(a-b)<b$ or when $(a-b)\geq b$. In the first case $(a-b)<b$ we get $(b,a-b)<(a,b)$. We define new coefficients $a':=b$, $b':=a-b$, $c':=d$ and $d':=c+d-b$. Otherwise if $(a-b)\geq b$, then $(a-b,b)<(a,b)$, and we define $a':=a-b$, $b':=b$, $c':=c+d-b$, and $d':=d$. For both cases, the new coefficients fulfill the inductive hypothesis, because:
Thanks to $bc+ad\geq ab$, we have

$$\begin{aligned} b'c'+a'd' &= (a-b)d+b(c+d-b)\\ &= ad+bc-b^2\\ &\geq ab-b^2=(a-b)b\\ &= a'b'. \end{aligned}$$

By induction, the self-intersection of the new linear system $\widehat{M}$ is equal to

$$\widehat{M}^2=(M^2-b^2)-b(a-b)=M^2-ab.$$

□

**Lemma 4.27.** *Assume that $bc+ad\leq ab$. Then $M^2-\widehat{M}^2=ad+bc$.*

*Proof.* We prove the lemma by induction, once more on $(a,b)$, with $b\leq a$. Thus we consider the lexicographic order $\leq$ on $\Delta^{\geq}$, as we have done in the proof of Lemma 4.26.

Suppose that $(a,b)=0$. Then $M$ is base-point free and so $\widehat{M}=M^2=M^2-(0d+0c)$. Now suppose that the statement is true for $(a',b')<(a,b)$. Our aim is to prove it for $(a,b)$. We blow up the base-point $p$, take the pullback of the divisors $D_i$, and remove the fixed part, which is the exceptional divisor $(c+d)E$ of the blowup, if $c+d\leq b$, or the divisor $bE$, otherwise. Hence we need to distinguish two cases.

Let us suppose first that $c+d\leq b$ $(\leq a)$. Restricted to the preimage of our neighborhood of $p$, the divisors are

$$a\widehat{H}+(a-(c+d))E, \qquad b\widehat{K}+(b-(c+d))E \qquad \text{and} \qquad c\widehat{H}+d\widehat{K}.$$

Here, $\widehat{H}$ and $\widehat{K}$ are the strict transforms of $H$ and $K$. Let $\widehat{M}$ be the linear system generated by these three divisors, then $\widehat{M}^2=M^2-(c+d)^2$. On the preimage, the linear system $\widehat{M}$ has precisely two new base-points: the intersection points of $\widehat{H}$ and $\widehat{K}$ with $E$. Locally near these points the three divisors spanning $\widehat{M}$ are respectively

$$a\widehat{H}+(a-(c+d))E, \qquad (b-(c+d))E \qquad \text{and} \qquad c\widehat{H},$$

and

$$(a-(c+d))E, \qquad b\widehat{K}+(b-(c+d))E \qquad \text{and} \qquad d\widehat{K}.$$

We claim that for both points the coefficients of these three divisors satisfy the assumption of Lemma 4.26.
Let us verify it for the first point $\widehat{H} \cap E$: if $c \geq (b-(c+d))$, then define $a' := c$, $b' := b-(c+d)$, $c' := a$, and $d' := a-(c+d)$, otherwise define $a' := b-(c+d)$, $b' := c$, $c' := a-(c+d)$, and $d' := a$. For both the cases $d' \geq b'$ so that $b'c' + a'd' \geq a'd' \geq a'b'$.
Regarding the second point $\widehat{K} \cap E$, we have: if $d \geq (a-(c+d))$, then define $a' := d$, $b' := a-(c+d)$, $c' := b$, and $d' := b-(c+d)$, otherwise define $a' := a-(c+d)$, $b' := d$, $c' := b-(c+d)$, $d' := b$. In the first case $c' \geq a'$, while in the second case $d' \geq b'$. Therefore we get $b'c' + a'd' \geq a'b'$ for both cases.
Thus Lemma 4.26 applies for both points and the self-intersection of the new linear system $\widehat{M}$ at the final step is amount to

$$\widehat{M}^2 = (M^2 - (c+d)^2) - (b-(c+d))c - (a-(c+d))d = M^2 - (ad+bc).$$

It remains to discuss the case $c + d \geq b$.
As we have already done before, we blow up the base-point $p$, take the pullback of the divisors $D_i$, and remove the fixed part, which this time is the exceptional divisor $bE$ of the blowup. Restricted to the preimage of our neighborhood of $p$, these divisors are:

$$a\widehat{H} + (a-b)E, \qquad b\widehat{K} \qquad \text{and} \qquad c\widehat{H} + d\widehat{K} + (c+d-b)E.$$

Here $\widehat{M}^2 = M^2 - b^2$. If $b = 0$ or $a = b$, then $\widehat{M}$ is base-point free. In the first case $b = 0$, we get $ad = bc + ad \leq ab = 0$, so $\widehat{M}^2 = M^2 - b^2 = M^2 = M^2 - (ad+bc)$, and we are done. In the second case $a = b$, we get, thanks to the assumptions $ad + bc \leq ab$ and $b \leq c + d$, that

$$\begin{aligned} a(c+d) &= ad + bc \\ &\leq ab \\ &\leq a(c+d) \end{aligned}, \qquad \text{so} \qquad c + d = b = a.$$

Also in this case we are done, because $\widehat{M}^2 = M^2 - b^2 = M^2 - (ad+bc)$.
It remains to consider when $a - b = 0$ or $b = 0$ does not hold. In this case, on the preimage, the linear system $\widehat{M}$ would have precisely one new base-point, the intersection point of $\widehat{K}$ and $E$. Locally near this point the three divisors spanning $\widehat{M}$ are:

$$(a-b)E, \qquad b\hat{K} \qquad \text{and} \qquad d\hat{K} + (c+d-b)E.$$

We need to distinguish two cases, when $(a-b) < b$ or when $(a-b) \geq b$. In the first case $(a-b) < b$ we get $(b, a-b) < (a, b)$. We define new coefficients $a' := b$, $b' := a-b$, $c' := d$ and $d' := c+d-b$. Otherwise if $(a-b) \geq b$, then $(a-b, b) < (a, b)$, and we define $a' := a-b$, $b' := b$, $c' := c+d-b$, and $d' := d$. For both cases, the new coefficients fulfill the inductive hypothesis, because:

Thanks to $bc + ad \leq ab$, we have

$$\begin{aligned} b'c' + a'd' &= (a-b)d + b(c+d-b) \\ &= ad + bc - b^2 \\ &\leq ab - b^2 = (a-b)b \\ &= a'b'. \end{aligned}$$

By induction, the self-intersection of the new linear system $\widehat{M}$ is equal to

$$\begin{aligned} \widehat{M}^2 &= (M^2 - b^2) - (a'd' + b'c') \\ &= M^2 - b^2 - (ad + bc - b^2) \\ &= M^2 - (ad + bc). \end{aligned}$$

□

By applying Lemma 4.26 and Lemma 4.27 it follows directly

**Theorem 4.28** (Correction Term Formula)**.** *Let $M$ be a two-dimensional linear system on a surface $S$ spanned by $D_1$, $D_2$, and $D_3$. Assume that $M$ has only isolated base-points, smooth for $S$, and that in a neighborhood of a basepoint $p$ we can write the divisors $D_i$ as*

$$D_1 = aH, \quad D_2 = bK \quad \textit{and} \quad D_3 = cH + dK.$$

*Here $H$ and $K$ are reduced, smooth, and intersect transversally at $p$ and $a, b, c, d$ are non-negative integers, $b \leq a$. Let $\widehat{M}$ be the strict transform of $M$ along $p$. Then*

$$M^2 - \widehat{M}^2 = \min \{ab, \ ad + bc\}.$$

4.6. **Example of the computation of the degree of the canonical map.** In this section we give an example how to compute the degree of the canonical of a regular product-quotient surface of geometric genus three, whenever Property (#) holds. In addition, in this way we also show the main steps in the operation of the MAGMA script for calculating the degree of the canonical map.

Let us consider the family of surfaces no. 1. in [18, Thm 2.3], which have degree of the canonical map 18.

Surfaces $S$ of no.1 of [18, Thm 2.3] can be described by the following pair of spherical systems of generators of the group $G = S_3 \times \mathbb{Z}_3^2$:

| | $q_1$ | $q_2$ | $q_3$ |
|---|---|---|---|
| branch point | $(1:1)$ | $(0:1)$ | $(-1:1)$ |
| generator | $g_1 := (\tau, (1,0))$ | $g_2 := (\sigma^2, (2,2))$ | $g_3 := (\sigma\tau, (0,1))$ |

| | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|---|---|---|---|---|
| branch point | $(1:1)$ | $(0:1)$ | $(1:\lambda)$ | $(-1:1)$ |
| generator | $h_1 := (\sigma\tau, 0)$ | $h_2 := (\sigma, (1,0))$ | $h_3 := (\mathrm{Id}, (1,1))$ | $h_4 := (\tau, (1,2))$ |

,

Here $\sigma$ and $\tau$ are a rotation (3-cycle) and a reflection (transposition) of the group $S_3$ respectively.

Instead, $q_i$ are the branch points of the pair of $G$-coverings $C_i \to \mathbb{P}^1$ defining $S$, and the respective generator of $q_i$ in the tables is the local monodromy of a point over $q_i$.

Notice that the second covering $C_2 \to \mathbb{P}^1$ depends from one parameter $\lambda$, with $\lambda \neq -1, 1$ since $C_2$ is smooth.

Consider the three irreducible characters of $S_3$, so the trivial character 1, the character *sgn* computing the sign of a permutation, and the only 2-dimensional irreducible character $\mu := \frac{1}{2}(\chi_{reg} - sgn - 1)$, where $\chi_{reg}$ is the character of the regular representation of $S_3$.
Let us also fix a basis $e_1, e_2$ of $\mathbb{Z}_3^2$ and consider the dual characters $\epsilon_1$, $\epsilon_2$ of $e_1$ and $e_2$, i.e. the characters defined by

$$\epsilon_i(ae_1 + be_2) := \zeta_3^{a\delta_{1i}+b\delta_{2i}}, \qquad \zeta_3 := e^{\frac{2\pi i}{3}},$$

where $\delta_{ij}$ is the Kronecker delta.

We apply Chevalley-Weil formula [24, Thm. 1.3.3] to both the curves $C_1$ and $C_2$ defining $S$ to compute the canonical characters $\chi^1_{can}$ and $\chi^2_{can}$ respectively:

$$\chi^1_{can} = \epsilon_1^2 \cdot \epsilon_2^2 + sgn \cdot \epsilon_1 \cdot \epsilon_2 + sgn \cdot \epsilon_2 + sgn \cdot \epsilon_1 + \mu \cdot \epsilon_1 \cdot \epsilon_2 + \mu \cdot \epsilon_1^2 \cdot \epsilon_2 + \mu \cdot \epsilon_1 \cdot \epsilon_2^2;$$
$$\chi^2_{can} = sgn \cdot \epsilon_1^2 \cdot \epsilon_2 + sgn \cdot \epsilon_1^2 \cdot \epsilon_2^2 + sgn \cdot \epsilon_1 \cdot \epsilon_2 + sgn \cdot \epsilon_1 + sgn \cdot \epsilon_2^2 + \mu \cdot \epsilon_1 + \mu \cdot \epsilon_2 + 2\mu \cdot \epsilon_2^2 + sgn \cdot \epsilon_1^2 + \epsilon_1^2 + \mu \cdot \epsilon_1^2 + \mu \cdot \epsilon_1 \cdot \epsilon_2,$$

We notice that the irreducible characters $\chi$ such that $\chi$ occurs on $\chi^1_{can}$ and $\overline{\chi}$ occurs on $\chi^2_{can}$ have degree one, so Property (#) is satisfied. These characters are precisely:

$$sgn \cdot \epsilon_1 \cdot \epsilon_2, \qquad sgn \cdot \epsilon_2, \qquad \text{and} \qquad sgn \cdot \epsilon_1.$$

From Theorem 4.15 we have $H^{2,0}(S) = \left(H^{1,0}(C_1) \otimes H^{1,0}(C_2)\right)^{S_3 \times \mathbb{Z}_3^2}$ decomposes into three pieces of dimension one:

$$H^{1,0}(C_1)^{sgn\cdot\epsilon_1\cdot\epsilon_2} \otimes H^{1,0}(C_2)^{sgn\cdot\epsilon_1^2\cdot\epsilon_2^2}, \quad H^{1,0}(C_1)^{sgn\cdot\epsilon_2} \otimes H^{1,0}(C_2)^{sgn\cdot\epsilon_2^2},$$
$$H^{1,0}(C_1)^{sgn\cdot\epsilon_1} \otimes H^{1,0}(C_2)^{sgn\cdot\epsilon_1^2}.$$

Corollary 4.23 determines which is respectively a generator of the associated linear subsystem given by each of these pieces:

$$R^{vert}_{(0,1)} + R^{hor}_{(1,\lambda)} + 2R^{hor}_{(-1,1)},$$
$$2R^{vert}_{(1,1)} + 2R^{hor}_{(0,1)},$$
$$2R^{vert}_{(-1,1)} + 4R^{hor}_{(-1,1)}.$$

Thus, the above three divisors are spanning the linear system $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$.

Notice then $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$ has not fixed part, so that

$$M^2 = \left(2R^{vert}_{(1,1)} + 2R^{hor}_{(0,1)}\right)^2 = 4 \cdot 2 \cdot \left(R^{vert}_{(1,1)} \cdot R^{hor}_{(0,1)}\right) = 8\frac{54}{6} \cdot \frac{54}{3} = 24 \cdot 54.$$

Furthermore, $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$ has precisely 81 (non reduced) isolated base-points $R^{vert}_{(1,1)} \cap R^{hor}_{(-1,1)}$. We can compute $M^2 - \widehat{M}^2$ by applying *Correction term formula*

4.28, recursively for each base-point of $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$. Indeed, in a neighbourhood of each of these base-points the three divisors are respectively

$$2R^{hor}_{(-1,1)}, \qquad 2R^{vert}_{(1,1)} \qquad \text{and} \qquad 4R^{hor}_{(-1,1)},$$

and since $R^{vert}_{(1,1)}$ and $R^{hor}_{(-1,1)}$ are transversal, then we are in the situation of Theorem 4.28, with $H = R^{hor}_{(-1,1)}$ and $K = R^{vert}_{(1,1)}$, $a = 4, b = c = 2$, and $d = 0$. This implies $ad + bc = 4 \leq ab = 8$. The correction term is $ab + cd = 4$ for each of the 81 base-points. Thus

$$M^2 - \widehat{M}^2 = 4\cdot 81.$$

The degree of the canonical map is therefore given by

$$\deg(\Phi_{K_S}) = \frac{1}{54}\widehat{M}^2 = \frac{1}{54}\left(M^2 - (M^2 - \widehat{M}^2)\right) = \frac{1}{54}(54\cdot 24 - 4\cdot 81) = 18.$$

## 5. Some remarks on the computational complexity

In this section we discuss the efficiency of the algorithm described in Subsection 2.1.
As already said in the previous sections, we adopt the computational algebra system MAGMA [13] although all the calculations can be done through others computational algebra programs with a database of finite groups, such as GAP4.
Firstly, we compare the efficiency of our script with respect to the previous versions developed in [1], [2], [8], [7], and [23].
Thus, in the following table we firstly report the computation time and memory usage of *FindSurfaces_with_Fixed_Ksquare_chi(ExistingSurfaces(ListGroups(1,$K^2$)))* for each $K^2 \in \{-1,\dots,8\}$, which returns automatically all regular surfaces with $\chi(\mathcal{O}_S) = 1$ and $K_S^2 = K^2$ that are product-quotient surfaces:

| $K^2$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|
| time ($s$) | 332.79 | 0.0 | 712.28 | 111.93 | 1360.74 | 917.95 | 1268.29 |
| memory (MB) | 216.88 | 32.09 | 184.78 | 184.75 | 184.78 | 184.75 | 216.75 |

Table 12

| $K^2$ | 1 | 0 | $-1$ |
|---|---|---|---|
| time ($s$) | 812.59 | 15153.71 | 343947.72 |
| memory (MB) | 216.78 | 325.72 | 431 |

Table 13

**Remark 5.1.** We compared our results with respect to those of [1] and [2] (for $K^2 = 8$) and those listed in the tables of [8] (for $1 \leq K^2 \leq 8$). We noticed that there are two mistakes, since the authors forgot the possibly exchanging of the factors which provides only one irreducible family of surfaces instead of two, so $N = 1$, in the cases $G = \mathbb{Z}_5^2$ and $G = \mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$.

The mistake found for $G = \mathbb{Z}_5^2$ was already mentioned in [5, Remark 3.2 (3)], while that for $G = \mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$ seems never discovered to our knowledge.

**Remark 5.2.** Comparing the results for $K^2 = 0$ with respect those of [7], we noticed that [7, Table 1] does not contain the following two other cases:

| $Sing(X)$ | $t_1$ | $t_2$ | $Id(G)$ |
|---|---|---|---|
| $1/4,\ 1/2^4,\ 3/4$ | $2,4,6$ | $2,4,6$ | $\langle 72,40\rangle$ |
| $1/4,\ 1/2^4,\ 3/4$ | $2,4,5$ | $2,4,6$ | $\langle 120,34\rangle$ |

TABLE 14

We verified that the MAGMA script of [7] returns also these results, so the authors just forgot to include them in their list of Table 1.

We point out also that our code returns other three results than those of [7, Table 1] and 14, listed in Table 15. These cases were not listed in Table 1 of

| $Sing(X)$ | $t_1$ | $t_2$ | $Id(G)$ |
|---|---|---|---|
| $2/5,\ 1/2^4,\ 3/5$ | $2,4,5$ | $2,4,5$ | $\langle 160,234\rangle$ |
| $1/3^2,\ 1/2^2,\ 2/3^2$ | $3,3,4$ | $3,3,4$ | $\langle 48,3\rangle$ |
| $1/3^2,\ 1/2^2,\ 2/3^2$ | $3,3,4$ | $2,3,7$ | $\langle 168,42\rangle$ |

TABLE 15

[7] since they do not provide surfaces of general type. Indeed, the number $\xi(X)$ is respectively equal to $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{2}{5}$ for such cases, so that $\xi(X) < \frac{1}{2}$ and by [7, Thm 5.3 and Cor 5.4] they cannot give surfaces of general type.

We also excluded manually the secondary output of *ListGroups*$(0,1)$ (with a similar approach such as that explained in Section 3 for the case $(K^2, \chi) = (32, 4)$) to prove the following

**Theorem 5.3.** *Let $S$ be a product-quotient surface with $K_S^2 = p_g(S) = q(S) = 0$, then one of the following holds:*

*(1) $S$ realizes one of the families of surfaces described in* [7, Table 1]*, Table 14, Table 15. Furthermore, all these surfaces are minimal and not of general type;*

*(2) $S$ is the surface described in* [7, Prop 7.1]*. In particular, it is a surface of general type whose minimal model is a numerical Godeaux surface with torsion of order* 4.

**Remark 5.4.** Regarding the classification obtained for $K^2 = -1$, we get one case more than those two found in [7], see Table 16. This happened because the script developed in [7] looks for only surfaces of general type and so automatically exclude cases with $\xi(X) < \frac{1}{2}$. However, the last case found by us has $\xi(X) = \frac{2}{5}$ and so has been automatically excluded.

| $Sing(X)$ | $t_1$ | $t_2$ | $Id(G)$ |
|---|---|---|---|
| $1/5,\ 2/5^2, 4/5$ | $2,5,5$ | $3,3,5$ | $\langle 60,5\rangle$ |
| $1/5,\ 1/2^4,\ 4/5$ | $2,4,5$ | $2,4,5$ | $\langle 160,234\rangle$ |
| $1/5^5$ | $5,5,5$ | $5,5,5$ | $\langle 25,2\rangle$ |

TABLE 16

Furthermore, we found two irreducible families sharing the same algebraic data of the group $\mathbb{Z}_5^2$ instead of only one family found in [7]. We have also excluded manuallly the secondary output of *ListGroups*$(-1,1)$ to prove the following

**Theorem 5.5.** *Let $S$ be a product-quotient surface with $K_S^2 = -1$, $p_g(S) = q(S) = 0$, then $S$ realizes one of the families of surfaces described in Table 16. Furthermore, the first two cases of the table give product-quotient surfaces that are minimal and not of general type. Instead, the last case with group $\mathbb{Z}_5^2$ gives two irreducible families of surfaces that are not minimal and whose minimal model is a numerical Godeaux surface with torsion of order 5.*

Regarding the efficiency of our script to obtain the results of [24], we see that *FindSurfaces_with_Fixed_Ksquare_chi(ExistingSurfaces(ListGroups($K^2$,$\chi$)))* for $K^2 = 16$ and $\chi = 2$ has a computation time equal to 27306.09 seconds, whilst the memory usage consists of 825 megabyte.

It remains to discuss the computational complexity of our program for the classification given in Section 3 of the present paper. In particular, we only give here details for the case $\chi = 4$ and $K^2 = 32$:

(1) *ListGroup(32,4)* has a computation time equal to 6075.11 seconds, and a memory usage of 325.78 megabyte;
(2) The function *ExistingSurfaces* computed on the main output of *ListGroup(32,4)* has a computation time equal to 587.860 seconds and a memory usage of 182.66 megabyte;
(3) The function *FindSurfaces_with_Fixed_Ksquare_chi* computed on the list produced by *ExistingSurfaces* requires 135494.74 seconds and 49756.03 megabyte.

**Remark 5.6.** The computation at (3) is done by excluding from the list produced by *ExistingSurfaces* six triples, which make the computation time too long.

They are triples *no.*$9, 15, 23, 35, 67, 72$ of tables 17 and 18 of the appendix of this paper.

We ran *FindSurfaces_with_Fixed_Ksquare_chi* for each of these triples separately. However, the computer used could not handle the extensive calculations and terminated the program automatically for triples *no.* 15, 35, and 67. As a result, we are only able to confirm the existence of irreducible families of product-quotient surfaces with compatible algebraic data for these triples, but we cannot compute the exact number of them. Therefore, we marked an interrogative point in the $N$-column for each of these triples. Similar difficulties were encountered for other challenging cases with $K^2$ ranging from 23 to 30, where we consistently used the symbol '?' in the $N$-column.

Instead,

- for *no.* 9 the computation time is 341875.6 seconds and it requires 5893.69 megabyte;
- for *no.* 23 we need 137615.33 seconds and it requires 214.19 megabyte;
- for *no.* 72 the computation time is 56131.82 seconds and it requires 214.19 magabyte.

Let us give a brief comment on the computational complexity of the hardest steps to prove Theorem 3.2 of Section 3.

To attain Proposition 3.5 we selected those triples of the secondary output of *ListGroups* having a group order different from 1024, 1536, and less or equal to 2000. We used *HowToExclude* script to exclude such list of triples. The computation time has been 22335.810 seconds and it has required 265,56 megabyte.

Instead, to exclude those triples having a group order equal to 1536, that are in total five, we always run separately *HowToExclude* for each of them. The computation time for each of them was approximately of 162000 seconds (45 hours) and the memory usage of 740 megabyte.

Regarding Proposition 3.11, we encountered difficulties to prove that spherical systems of generators of a group of order 1536 with signature $[4,4,4]$ do not exist. The function *ExSphSystem* has required more or less one week of computation and a memory usage of 1072.44 megabytes.

## Appendix A.

In this appendix we list all regular product-quotient surfaces of general type with $23 \leq K^2 \leq 32$ and $p_g = 3$. In particular, we list the following information in the columns of tables 17 to 29:

- $K_S^2$ is the self-intersection of the canonical class of $S$;
- $G$ is the group, and $Id$ is the identifier of the group in the MAGMA database of small groups; hence the pair $\langle d, n\rangle$ of each row denotes that $G$ is the $n$-th group of order $d$ in the MAGMA database of small groups. Whenever $G$ has not an easy description, we simply denote it by $G(d, n)$, the group in the MAGMA database having identifier $\langle d, n\rangle$;
- $\mathrm{Sing}(X)$ is the singular locus of the quotient model $X := (C_1 \times C_2)/G$ defining the product-quotient surface $S$. It is given as a sequence of rational numbers with multiplicities, describing the types of cyclic quotient singularities. For instance, $3/5^4$ means 4 singular points of type $\frac{1}{5}(1,3)$;
- $t_1$ and $t_2$ are the signatures of the corresponding spherical systems of generators, cf. Definition 1.4;
- $N$ is the number of irreducible families. Indeed our tables have 555 lines, but we collect in the same line $N$ families, which share all the other data. We employ the symbol ? whenever we are unable to determine the exact number of families in a row due to computational time constraints or machine memory overflow;

- $\deg(\Phi_S)$ contains, for each family of the row, the degree of the canonical map of a surface $S$ belonging to that family, whenever the computation of the degree can be done. For example, if there are $N$ irreducible families in a row, where $N = 3$, and the degrees listed in the $\deg(\Phi_S)$ box for that row are 12 and 16, it indicates that the degree of the canonical map has been computed for surfaces from only two of the three families. Specifically, the degree is 12 for one family and 16 for the other.

  Furthermore, since the degree of the canonical map is not a topological invariant then it may happen that surfaces belonging to the same family have distinct degrees of the canonical map. In this case, we simply list sequentially all degrees of the canonical map of the surfaces belonging to that family. For instance, suppose $\deg(\Phi_S)$ of a row is $12, (18, 16), 18$. This means surfaces of two of these three families have a degree of the canonical map that is constant on the family and equal respectively to 12 and 18, while the other family has surfaces with a degree of the canonical map either equal to 18 or 16.

  The number 0 means that the image of $\Phi_S$ has dimension 1.

For the groups occurring in tables 17 to 29 we use the following notation:
$\mathbb{Z}_n^k$ is $k$-times the cyclic group of order $n$.
$S_n$ is the symmetric group of $n$ letters.
$\mathcal{A}_n$ is the alternating group.
$\mathrm{ASL}(n, k)$ is the affine special linear group of $\mathbb{Z}_k^n$.
$\mathrm{PSL}(2, n)$ is the group of $2 \times 2$ matrices over $\mathbb{Z}_n$ with determinant 1 modulo the subgroup generated by $-\mathrm{Id}$.
$\mathrm{SO}(3, 7)$ is the group of $3 \times 3$ orthogonal matrices over $\mathbb{F}_7$ with determinant 1.
$\mathrm{He}p$ is the Heisenberg group of order $p^3$:

$$\mathrm{He}p := \langle x, y, z | z^{-1}xyx^{-1}y^{-1}, x^p, y^p, z^p, xz = zx, yz = zy\rangle$$

A 3-dimensional representation of $\mathrm{He}p$ (over the field $\mathbb{Z}_p$) is given by sending

$$x \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$Q_8$ is the quaternion group:

$$Q_8 := \langle x, y | x^4, x^2y^{-2}, y^{-1}xyx\rangle.$$

$K \wr H$ is the wreath product, so it is the semidirect product $K^H \rtimes H$, where $K^H$ is the set of *functions* $f \colon H \to K$, with a group operation given by pointwise multiplication. Here $H$ is acting on $K^H$ via left multiplication:

$$h \cdot f := f \circ h^{-1}, \qquad f \colon H \to K \in K^H.$$

| *no.* | $K_S^2$ | Sing($X$) | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg($\Phi_S$) |
|---|---|---|---|---|---|---|---|---|
| 1 | 32 | | $2^6$ | $2^8$ | $\mathbb{Z}_2^3$ | $\langle 8, 5\rangle$ | 3 | $8, 16^2$ |
| 2 | 32 | | $2^5$ | $2^{12}$ | $\mathbb{Z}_2^3$ | $\langle 8, 5\rangle$ | 3 | 0,4,8 |
| 3 | 32 | | $3^4$ | $3^7$ | $\mathbb{Z}_3^2$ | $\langle 9, 2\rangle$ | 2 | 6, 12 |
| 4 | 32 | | $3^5$ | $3^5$ | $\mathbb{Z}_3^2$ | $\langle 9, 2\rangle$ | 1 | 9 |
| 5 | 32 | | $2^3, 4^2$ | $2^3, 4^2$ | $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$ | $\langle 16, 3\rangle$ | 2 | 16 |
| 6 | 32 | | $2^2, 4^2$ | $2^2, 4^4$ | $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$ | $\langle 16, 3\rangle$ | 2 | |
| 7 | 32 | | $2^2, 4^2$ | $2^5, 4^2$ | $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$ | $\langle 16, 3\rangle$ | 6 | 8 |
| 8 | 32 | | $2^5$ | $2^5, 4^2$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 4 | |
| 9 | 32 | | $2^3, 4$ | $2^{12}$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 6 | 0 |
| 10 | 32 | | $2^3, 4^2$ | $2^6$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 2 | |
| 11 | 32 | | $2^2, 4^4$ | $2^5$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 1 | |
| 12 | 32 | | $2^6$ | $2^6$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 1 | 32 |
| 13 | 32 | | $2^5$ | $2^8$ | $\mathbb{Z}_2^4$ | $\langle 16, 14\rangle$ | 13 | $0, 8^5, 16^7$ |
| 14 | 32 | | $2^6$ | $2^6$ | $\mathbb{Z}_2^4$ | $\langle 16, 14\rangle$ | 6 | $8, 16^3, 32^2$ |
| 15 | 32 | | $2^{12}$ | $3, 4^2$ | $S_4$ | $\langle 24, 12\rangle$ | ? | |
| 16 | 32 | | $2^4, 3$ | $4^4$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 17 | 32 | | $2, 3, 4^2$ | $2^6$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 18 | 32 | | $2^2, 3^2$ | $2^2, 4^4$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 19 | 32 | | $2^5$ | $2^5, 6$ | $\mathbb{Z}_2^2 \times S_3$ | $\langle 24, 14\rangle$ | 1 | |
| 20 | 32 | | $2^2, 4^2$ | $4^4$ | $G(32, 6)$ | $\langle 32, 6\rangle$ | 1 | |
| 21 | 32 | | $2^2, 4^2$ | $2^3, 4^2$ | $G(32, 22)$ | $\langle 32, 22\rangle$ | 7 | 16 |
| 22 | 32 | | $2^2, 4^4$ | $2^3, 4$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 2 | |
| 23 | 32 | | $2^3, 4$ | $2^5, 4^2$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 30 | |
| 24 | 32 | | $2^2, 4^2$ | $2^3, 4^2$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 1 | |
| 25 | 32 | | $2^3, 4^2$ | $2^5$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 4 | |
| 26 | 32 | | $2^2, 4^2$ | $2^6$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 4 | |
| 27 | 32 | | $2^2, 4^4$ | $2^3, 4$ | $\mathbb{Z}_2^2 \rtimes D_4$ | $\langle 32, 28\rangle$ | 1 | |
| 28 | 32 | | $2^5$ | $2^6$ | $\mathbb{Z}_2^2 \times D_4$ | $\langle 32, 46\rangle$ | 4 | 24 |
| 29 | 32 | | $2^3, 4^2$ | $2^5$ | $\mathbb{Z}_2^2 \times D_4$ | $\langle 32, 46\rangle$ | 2 | |
| 30 | 32 | | $2^3, 4^2$ | $2^5$ | $Q_8 \rtimes \mathbb{Z}_2^2$ | $\langle 32, 49\rangle$ | 1 | |
| 31 | 32 | | $2^2, 4, 12$ | $2^2, 4^2$ | $D_6 \rtimes \mathbb{Z}_4$ | $\langle 48, 14\rangle$ | 1 | |
| 32 | 32 | | $2^2, 4^4$ | $3, 4^2$ | $\mathcal{A}_4 \rtimes \mathbb{Z}_4$ | $\langle 48, 30\rangle$ | 3 | |
| 33 | 32 | | $2^3, 4$ | $2^5, 6$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 1 | |
| 34 | 32 | | $4^2, 6$ | $2^6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 3 | |
| 35 | 32 | | $2, 4, 6$ | $2^{12}$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 36 | 32 | | $2^2, 4^2$ | $2^4, 3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 37 | 32 | | $2^2, 4^2$ | $2^2, 6^2$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 38 | 32 | | $2^2, 4^4$ | $2^3, 3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 4 | |
| 39 | 32 | | $2^3, 6$ | $4^4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 40 | 32 | | $2^3, 4^2$ | $2^3, 6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 41 | 32 | | $2, 3, 4^2$ | $2^5$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 42 | 32 | | $7^3$ | $7^3$ | $\mathbb{Z}_7^2$ | $\langle 49, 2\rangle$ | 7 | $0, 5, 7, 10, 11, 14^2$ |

TABLE 17. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 = 32$

| *no.* | $K_S^2$ | Sing($X$) | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg($\Phi_S$) |
|---|---|---|---|---|---|---|---|---|
| 43 | 32 | | $2, 5^2$ | $3^7$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 2 | |
| 44 | 32 | | $2^8$ | $3^2, 5$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 1 | |
| 45 | 32 | | $2^4, 3$ | $5^3$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 1 | |
| 46 | 32 | | $3^4$ | $5^3$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 1 | |
| 47 | 32 | | $2^6$ | $3, 5^2$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 2 | |
| 48 | 32 | | $2^2, 4^2$ | $2^2, 4^2$ | $G(64, 60)$ | $\langle 64, 60\rangle$ | 3 | 32 |
| 49 | 32 | | $2^2, 4^2$ | $2^2, 4^2$ | $\mathbb{Z}_4 \rtimes (\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4)$ | $\langle 64, 71\rangle$ | 1 | |
| 50 | 32 | | $2^3, 4$ | $2^6$ | $G(64, 73)$ | $\langle 64, 73\rangle$ | 1 | |
| 51 | 32 | | $2^3, 4$ | $2^3, 4^2$ | $G(64, 73)$ | $\langle 64, 73\rangle$ | 4 | |
| 52 | 32 | | $2^2, 4^2$ | $2^2, 4^2$ | $G(64, 75)$ | $\langle 64, 75\rangle$ | 1 | |
| 53 | 32 | | $2^3, 4$ | $4^4$ | $\mathbb{Z}_2 \wr \mathbb{Z}_2^2$ | $\langle 64, 138\rangle$ | 1 | |
| 54 | 32 | | $2^3, 4$ | $2^3, 4^2$ | $\mathbb{Z}_2 \wr \mathbb{Z}_2^2$ | $\langle 64, 138\rangle$ | 6 | |
| 55 | 32 | | $2^5$ | $2^5$ | $G(64, 211)$ | $\langle 64, 211\rangle$ | 1 | |
| 56 | 32 | | $2^5$ | $2^5$ | $\mathbb{Z}_2^2 \times D_8$ | $\langle 64, 250\rangle$ | 1 | |
| 57 | 32 | | $2^2, 4, 12$ | $2^3, 4$ | $\mathbb{Z}_2^2 \rtimes D_{12}$ | $\langle 96, 89\rangle$ | 1 | |
| 58 | 32 | | $2^2, 4^2$ | $4^2, 6$ | $\mathrm{GL}(2, \mathbb{Z}_4)$ | $\langle 96, 195\rangle$ | 1 | |
| 59 | 32 | | $2, 4, 6$ | $2^2, 4^4$ | $\mathrm{GL}(2, \mathbb{Z}_4)$ | $\langle 96, 195\rangle$ | 10 | |
| 60 | 32 | | $2^2, 4^2$ | $2^3, 6$ | $\mathbb{Z}_2^2 \times S_4$ | $\langle 96, 226\rangle$ | 1 | |
| 61 | 32 | | $2^3, 4^2$ | $3, 4^2$ | $\mathbb{Z}_2^2 \rtimes S_4$ | $\langle 96, 227\rangle$ | 1 | |
| 62 | 32 | | $2^3, 3$ | $4^4$ | $\mathbb{Z}_2^2 \rtimes S_4$ | $\langle 96, 227\rangle$ | 3 | |
| 63 | 32 | | $2^6$ | $3, 4^2$ | $\mathbb{Z}_2^2 \rtimes S_4$ | $\langle 96, 227\rangle$ | 3 | |
| 64 | 32 | | $2^4, 5$ | $3, 4^2$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 65 | 32 | | $2, 5, 6$ | $4^4$ | $S_5$ | $\langle 120, 34\rangle$ | 2 | |
| 66 | 32 | | $2, 5, 6$ | $2^3, 4^2$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 67 | 32 | | $2, 4, 5$ | $2^2, 4^4$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160, 234\rangle$ | ? | |
| 68 | 32 | | $3, 7^2$ | $4^3$ | $\mathrm{PSL}(2, 7)$ | $\langle 168, 42\rangle$ | 4 | |
| 69 | 32 | | $3, 4^2$ | $7^3$ | $\mathrm{PSL}(2, 7)$ | $\langle 168, 42\rangle$ | 1 | |
| 70 | 32 | | $2^2, 4^2$ | $3^2, 7$ | $\mathrm{PSL}(2, 7)$ | $\langle 168, 42\rangle$ | 1 | |
| 71 | 32 | | $2^3, 4$ | $4^2, 6$ | $G(192, 955)$ | $\langle 192, 955\rangle$ | 1 | |
| 72 | 32 | | $2, 4, 6$ | $2^3, 4^2$ | $G(192, 955)$ | $\langle 192, 955\rangle$ | 7 | |
| 73 | 32 | | $2, 4, 6$ | $4^4$ | $G(192, 955)$ | $\langle 192, 955\rangle$ | 2 | |
| 74 | 32 | | $2, 6^2$ | $4^2, 10$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240, 189\rangle$ | 1 | |
| 75 | 32 | | $2, 4, 6$ | $2^2, 10^2$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240, 189\rangle$ | 1 | |
| 76 | 32 | | $4^3$ | $4^3$ | $G(256, 295)$ | $\langle 256, 295\rangle$ | 3 | |
| 77 | 32 | | $4^3$ | $4^3$ | $G(256, 298)$ | $\langle 256, 298\rangle$ | 2 | |
| 78 | 32 | | $4^3$ | $4^3$ | $G(256, 306)$ | $\langle 256, 306\rangle$ | 2 | |
| 79 | 32 | | $2, 6, 7$ | $2, 8^2$ | $\mathrm{SO}(3, 7)$ | $\langle 336, 208\rangle$ | 2 | |
| 80 | 32 | | $2, 3, 14$ | $2^2, 4^2$ | $\mathbb{Z}_2 \times \mathrm{PSL}(2, 7)$ | $\langle 336, 209\rangle$ | 1 | |
| 81 | 32 | | $2, 7, 14$ | $3, 4^2$ | $\mathbb{Z}_2 \times \mathrm{PSL}(2, 7)$ | $\langle 336, 209\rangle$ | 1 | |
| 82 | 32 | | $2, 6, 7$ | $4^3$ | $\mathbb{Z}_2 \times \mathrm{PSL}(2, 7)$ | $\langle 336, 209\rangle$ | 2 | |
| 83 | 32 | | $2, 6, 15$ | $3, 4^2$ | $\mathbb{Z}_3 \rtimes S_5$ | $\langle 360, 120\rangle$ | 1 | |
| 84 | 32 | | $2, 4, 6$ | $4^2, 10$ | $\mathbb{Z}_2^2 \rtimes S_5$ | $\langle 480, 951\rangle$ | 2 | |
| 85 | 32 | | $2, 3, 9$ | $7^3$ | $\mathrm{PSL}(2, 8)$ | $\langle 504, 156\rangle$ | 6 | |
| 86 | 32 | | $2, 5^2$ | $3^2, 11$ | $\mathrm{PSL}(2, 11)$ | $\langle 660, 13\rangle$ | 2 | |

TABLE 18. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 = 32$

| $no.$ | $K_S^2$ | Sing($X$) | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg($\Phi_S$) |
|---|---|---|---|---|---|---|---|---|
| 87 | 30 | $1/2^2$ | $2^3, 4$ | $2^{10}, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 6 | 0 |
| 88 | 30 | $1/2^2$ | $2^4, 4$ | $2^5, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 2 | 4 |
| 89 | 30 | $1/2^2$ | $2^3, 8$ | $2^5, 4$ | $\mathbb{Z}_2 \times D_8$ | $\langle 32, 39\rangle$ | 1 | |
| 90 | 30 | $1/2^2$ | $2^3, 12$ | $2^4, 4$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 1 | |
| 91 | 30 | $1/2^2$ | $2^3, 4$ | $2^3, 6, 12$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 1 | |
| 92 | 30 | $1/2^2$ | $2, 4, 6$ | $2^{10}, 4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 93 | 30 | $1/2^2$ | $2^2, 3, 4$ | $2^4, 4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 94 | 30 | $1/2^2$ | $2, 3^6$ | $2, 5^2$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 1 | |
| 95 | 30 | $1/2^2$ | $2^2, 4, 8$ | $2^3, 8$ | $(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$ | $\langle 64, 128\rangle$ | 2 | |
| 96 | 30 | $1/2^2$ | $2, 6, 12$ | $2^2, 3, 4$ | $S_3 \times S_4$ | $\langle 144, 183\rangle$ | 1 | |
| 97 | 30 | $1/2^2$ | $2, 7^3$ | $3^2, 4$ | PSL(2, 7) | $\langle 168, 42\rangle$ | 4 | |
| 98 | 30 | $1/2^2$ | $3^2, 4$ | $3^3, 6$ | ASL(2, 3) | $\langle 216, 153\rangle$ | 4 | |
| 99 | 30 | $1/2^2$ | $2, 4, 10$ | $2^2, 3, 4$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240, 189\rangle$ | 1 | |
| 100 | 30 | $1/2^2$ | $2, 9^2$ | $3^2, 6$ | $G(324, 160)$ | $\langle 324, 160\rangle$ | 3 | |
| 101 | 30 | $1/2^2$ | $2, 4, 7$ | $4, 6^2$ | $\mathbb{Z}_2 \times$ PSL(2, 7) | $\langle 336, 209\rangle$ | 2 | |
| 102 | 30 | $1/2^2$ | $2, 4, 5$ | $4, 6^2$ | $\mathbb{Z}_2 \times \mathcal{A}_6$ | $\langle 720, 766\rangle$ | 2 | |
| 103 | 29 | $1/3, 2/3$ | $2^{10}, 3$ | $3, 4^2$ | $S_4$ | $\langle 24, 12\rangle$ | ? | |
| 104 | 29 | $1/3, 2/3$ | $2^3, 4^2, 6$ | $3, 4^2$ | $\mathcal{A}_4 \rtimes \mathbb{Z}_4$ | $\langle 48, 30\rangle$ | 3 | |
| 105 | 29 | $1/3, 2/3$ | $3, 4^2$ | $4^4, 6$ | $\mathcal{A}_4 \rtimes \mathbb{Z}_4$ | $\langle 48, 30\rangle$ | 1 | |
| 106 | 29 | $1/3, 2/3$ | $2, 4, 6$ | $2^{10}, 3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 107 | 29 | $1/3, 2/3$ | $2^3, 3$ | $4^4, 6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 108 | 29 | $1/3, 2/3$ | $2^3, 3$ | $2^3, 4^2, 6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 4 | |
| 109 | 29 | $1/3, 2/3$ | $2, 4, 6$ | $4^4, 6$ | GL(2, $\mathbb{Z}_4$) | $\langle 96, 195\rangle$ | 1 | |
| 110 | 29 | $1/3, 2/3$ | $2, 4, 6$ | $2^3, 4^2, 6$ | GL(2, $\mathbb{Z}_4$) | $\langle 96, 195\rangle$ | 8 | |
| 111 | 29 | $1/3, 2/3$ | $2^3, 3, 4$ | $3, 4^2$ | $G(96, 227)$ | $\langle 96, 227\rangle$ | 3 | |
| 112 | 29 | $1/3, 2/3$ | $2, 3, 8$ | $4^4, 6$ | $G(192, 181)$ | $\langle 192, 181\rangle$ | 1 | |
| 113 | 29 | $1/3, 2/3$ | $2, 4, 6$ | $3, 4^3$ | $G(192, 955)$ | $\langle 192, 955\rangle$ | 2 | |
| 114 | 29 | $1/3, 2/3$ | $2^3, 3$ | $4, 6, 8$ | $G(192, 956)$ | $\langle 192, 956\rangle$ | 1 | |
| 115 | 29 | $1/3, 2/3$ | $2^3, 3$ | $4, 6, 8$ | $G(192, 1494)$ | $\langle 192, 1494\rangle$ | 1 | |
| 116 | 29 | $1/3, 2/3$ | $2, 4, 6$ | $2^2, 6, 10$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240, 189\rangle$ | 2 | |
| 117 | 29 | $1/3, 2/3$ | $2, 4, 6$ | $4, 6, 8$ | $G(384, 5602)$ | $\langle 384, 5602\rangle$ | 2 | |
| 118 | 29 | $1/3, 2/3$ | $2, 3, 10$ | $2, 4, 12$ | $G(1320, 133)$ | $\langle 1320, 133\rangle$ | 4 | |
| 119 | 28 | $1/2^4$ | $2^2, 4^2$ | $2^8, 4^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\langle 8, 2\rangle$ | 1 | 0 |
| 120 | 28 | $1/2^4$ | $2^5$ | $2^{11}$ | $\mathbb{Z}_2^3$ | $\langle 8, 5\rangle$ | 6 | $0^2, 4^3, 8$ |
| 121 | 28 | $1/2^4$ | $2^3, 4^3$ | $2^5$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 3 | |
| 122 | 28 | $1/2^4$ | $2^3, 4$ | $2^8, 4^2$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 5 | |
| 123 | 28 | $1/2^4$ | $2^3, 4$ | $2^{11}$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 14 | 0 |
| 124 | 28 | $1/2^4$ | $2^5$ | $2^6, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 6 | 8 |
| 125 | 28 | $1/2^4$ | $2^2, 3^2$ | $3^4, 6^2$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18, 3\rangle$ | 6 | $6^2$ |
| 126 | 28 | $1/2^4$ | $2^2, 3^5$ | $3, 6^2$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18, 3\rangle$ | 1 | |
| 127 | 28 | $1/2^4$ | $2^2, 3^2$ | $2^2, 3^5$ | $\mathbb{Z}_3 \rtimes S_3$ | $\langle 18, 4\rangle$ | 2 | |
| 128 | 28 | $1/2^4$ | $2^2, 3^2$ | $2^3, 4^3$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |

TABLE 19. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 \in \{30, 29, 28\}$

| *no.* | $K_S^2$ | Sing($X$) | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg($\Phi_S$) |
|---|---|---|---|---|---|---|---|---|
| 129 | 28 | $1/2^4$ | $2^{11}$ | $3, 4^2$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 130 | 28 | $1/2^4$ | $2^3, 6^2$ | $2^5$ | $\mathbb{Z}_2^2 \times S_3$ | $\langle 24, 14\rangle$ | 3 | |
| 131 | 28 | $1/2^4$ | $2^5$ | $2^5, 3$ | $\mathbb{Z}_2^2 \times S_3$ | $\langle 24, 14\rangle$ | 1 | |
| 132 | 28 | $1/2^4$ | $2, 4^2, 8$ | $2^2, 4^2$ | $\mathbb{Z}_4 \wr \mathbb{Z}_2$ | $\langle 32, 11\rangle$ | 1 | |
| 133 | 28 | $1/2^4$ | $2^3, 4$ | $2^3, 4^3$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 4 | |
| 134 | 28 | $1/2^4$ | $2^3, 4$ | $2^6, 4$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 30 | |
| 135 | 28 | $1/2^4$ | $2^3, 4$ | $2^3, 4^3$ | $\mathbb{Z}_2^2 \rtimes D_4$ | $\langle 32, 28\rangle$ | 4 | |
| 136 | 28 | $1/2^4$ | $2^4, 8$ | $2^5$ | $\mathbb{Z}_2 \times D_8$ | $\langle 32, 39\rangle$ | 2 | |
| 137 | 28 | $1/2^4$ | $2, 4^2, 8$ | $2^5$ | $\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$ | $\langle 32, 43\rangle$ | 1 | |
| 138 | 28 | $1/2^4$ | $2^2, 4, 6$ | $2^5$ | $\mathbb{Z}_2 \times D_{12}$ | $\langle 48, 36\rangle$ | 1 | |
| 139 | 28 | $1/2^4$ | $2^3, 4$ | $2^5, 3$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 1 | |
| 140 | 28 | $1/2^4$ | $2^2, 4, 6$ | $2^5$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 2 | |
| 141 | 28 | $1/2^4$ | $2^3, 4$ | $2^3, 6^2$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 2 | |
| 142 | 28 | $1/2^4$ | $2^2, 3, 4^2$ | $2^3, 4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 3 | |
| 143 | 28 | $1/2^4$ | $2^3, 3$ | $2^3, 4^3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 5 | |
| 144 | 28 | $1/2^4$ | $2^3, 4$ | $4^2, 6^2$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 145 | 28 | $1/2^4$ | $2, 4, 6$ | $2^{11}$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 146 | 28 | $1/2^4$ | $2, 4, 6$ | $2^8, 4^2$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 147 | 28 | $1/2^4$ | $2^2, 4, 6$ | $2^5$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 148 | 28 | $1/2^4$ | $2, 5^2$ | $2^2, 3^5$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 1 | |
| 149 | 28 | $1/2^4$ | $2^3, 4$ | $2^4, 8$ | $(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$ | $\langle 64, 128\rangle$ | 5 | |
| 150 | 28 | $1/2^4$ | $2, 4^2, 8$ | $2^3, 4$ | $D_4 \rtimes D_4$ | $\langle 64, 134\rangle$ | 1 | |
| 151 | 28 | $1/2^4$ | $2, 4^2, 8$ | $2^3, 4$ | $(\mathbb{Z}_4 \wr \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ | $\langle 64, 135\rangle$ | 1 | |
| 152 | 28 | $1/2^4$ | $2^3, 16$ | $2^5$ | $\mathbb{Z}_2 \times D_{16}$ | $\langle 64, 186\rangle$ | 1 | |
| 153 | 28 | $1/2^4$ | $2, 3^2, 4$ | $2^2, 3^2$ | $\mathbb{Z}_3 \rtimes S_4$ | $\langle 72, 43\rangle$ | 1 | |
| 154 | 28 | $1/2^4$ | $2^2, 4, 6$ | $2^3, 4$ | $\mathbb{Z}_2^2 \rtimes D_{12}$ | $\langle 96, 89\rangle$ | 1 | |
| 155 | 28 | $1/2^4$ | $2, 8, 12$ | $2^5$ | $(\mathrm{SL}(2,3) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ | $\langle 96, 193\rangle$ | 1 | |
| 156 | 28 | $1/2^4$ | $2, 4, 6$ | $2^3, 4^3$ | $\mathrm{GL}(2, \mathbb{Z}_4)$ | $\langle 96, 195\rangle$ | 9 | |
| 157 | 28 | $1/2^4$ | $2^2, 4, 6$ | $2^3, 4$ | $\mathbb{Z}_2^2 \times S_4$ | $\langle 96, 226\rangle$ | 2 | |
| 158 | 28 | $1/2^4$ | $2, 4, 5$ | $3^4, 6^2$ | $S_5$ | $\langle 120, 34\rangle$ | 2 | |
| 159 | 28 | $1/2^4$ | $2^3, 4$ | $5, 6^2$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 160 | 28 | $1/2^4$ | $2, 4, 6$ | $2^2, 5^3$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 161 | 28 | $1/2^4$ | $2^2, 5, 10$ | $2^3, 3$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 162 | 28 | $1/2^4$ | $2, 6, 10$ | $2^5$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 163 | 28 | $1/2^4$ | $2^3, 4$ | $2^3, 16$ | $G(128, 916)$ | $\langle 128, 916\rangle$ | 1 | |
| 164 | 28 | $1/2^4$ | $2, 4, 18$ | $2^5$ | $G(144, 109)$ | $\langle 144, 109\rangle$ | 1 | |
| 165 | 28 | $1/2^4$ | $2^2, 4, 6$ | $2^3, 3$ | $S_3 \times S_4$ | $\langle 144, 183\rangle$ | 1 | |
| 166 | 28 | $1/2^4$ | $2^2, 3, 12$ | $2^3, 3$ | $S_3 \times S_4$ | $\langle 144, 183\rangle$ | 1 | |
| 167 | 28 | $1/2^4$ | $2, 4, 5$ | $2^3, 4^3$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160, 234\rangle$ | ? | |
| 168 | 28 | $1/2^4$ | $2, 3, 8$ | $4^5$ | $G(192, 181)$ | $\langle 192, 181\rangle$ | 1 | |
| 169 | 28 | $1/2^4$ | $2, 5, 8$ | $3, 6^2$ | $\mathrm{SL}(2,5) \rtimes \mathbb{Z}_2$ | $\langle 240, 90\rangle$ | 1 | |
| 170 | 28 | $1/2^4$ | $2, 4, 6$ | $2^2, 5, 10$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240, 189\rangle$ | 1 | |

TABLE 20. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 = 28$

| $no.$ | $K_S^2$ | Sing$(X)$ | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg$(\Phi_S)$ |
|---|---|---|---|---|---|---|---|---|
| 171 | 28 | $1/2^4$ | $2,6,10$ | $2^3,4$ | $\mathbb{Z}_2\times S_5$ | $\langle 240,189\rangle$ | 2 | |
| 172 | 28 | $1/2^4$ | $2,4,8$ | $3,6^2$ | SO$(3,7)$ | $\langle 336,208\rangle$ | 2 | |
| 173 | 28 | $1/2^4$ | $2,4,8$ | $2,6^2$ | $\mathbb{Z}_2\times$ SO$(3,7)$ | $\langle 672,1254\rangle$ | 2 | |
| 174 | 28 | $1/2^4$ | $2,4,6$ | $2,8^2$ | $\mathbb{Z}_2\times$ SO$(3,7)$ | $\langle 672,1254\rangle$ | 2 | |
| 175 | 28 | $3/5^2$ | $2^3,5$ | $3^3,5$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 2 | |
| 176 | 28 | $3/5^2$ | $2^6,5$ | $3^2,5$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 1 | |
| 177 | 28 | $3/5^2$ | $2^3,5$ | $3,6,10$ | $\mathbb{Z}_2\times\mathcal{A}_5$ | $\langle 120,35\rangle$ | 1 | |
| 178 | 28 | $3/5^2$ | $2,4,5$ | $4^4,5$ | $\mathbb{Z}_2^4\rtimes D_5$ | $\langle 160,234\rangle$ | ? | |
| 179 | 28 | $3/5^2$ | $4^2,5$ | $4^2,5$ | $\mathbb{Z}_2^4\rtimes D_5$ | $\langle 160,234\rangle$ | 3 | |
| 180 | 28 | $3/5^2$ | $2,4,5$ | $2^3,4^2,5$ | $\mathbb{Z}_2^4\rtimes D_5$ | $\langle 160,234\rangle$ | ? | |
| 181 | 28 | $3/5^2$ | $2^3,5$ | $4^2,5$ | $\mathbb{Z}_2^4\rtimes D_5$ | $\langle 160,234\rangle$ | 3 | |
| 182 | 28 | $3/5^2$ | $2^3,5$ | $3^2,5$ | $\mathcal{A}_6$ | $\langle 360,118\rangle$ | 1 | |
| 183 | 28 | $3/5^2$ | $3^2,5$ | $4^2,5$ | $\mathcal{A}_6$ | $\langle 360,118\rangle$ | 2 | |
| 184 | 28 | $3/5^2$ | $2,4,5$ | $3^3,5$ | $\mathcal{A}_6$ | $\langle 360,118\rangle$ | 6 | |
| 185 | 28 | $3/5^2$ | $2,4,5$ | $3,6,10$ | $\mathbb{Z}_2\times\mathcal{A}_6$ | $\langle 720,766\rangle$ | 2 | |
| 186 | 27 | $1/5,4/5$ | $2^3,5$ | $3^3,5$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 2 | |
| 187 | 27 | $1/5,4/5$ | $2^6,5$ | $3^2,5$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 1 | |
| 188 | 27 | $1/5,4/5$ | $2^3,5$ | $3,6,10$ | $\mathbb{Z}_2\times\mathcal{A}_5$ | $\langle 120,35\rangle$ | 1 | |
| 189 | 27 | $1/5,4/5$ | $2,4,5$ | $4^4,5$ | $\mathbb{Z}_2^4\rtimes D_5$ | $\langle 160,234\rangle$ | 7 | |
| 190 | 27 | $1/5,4/5$ | $4^2,5$ | $4^2,5$ | $\mathbb{Z}_2^4\rtimes D_5$ | $\langle 160,234\rangle$ | 2 | |
| 191 | 27 | $1/5,4/5$ | $2,4,5$ | $2^3,4^2,5$ | $\mathbb{Z}_2^4\rtimes D_5$ | $\langle 160,234\rangle$ | ? | |
| 192 | 27 | $1/5,4/5$ | $2^3,5$ | $4^2,5$ | $\mathbb{Z}_2^4\rtimes D_5$ | $\langle 160,234\rangle$ | 3 | |
| 193 | 27 | $1/5,4/5$ | $2^3,5$ | $3^2,5$ | $\mathcal{A}_6$ | $\langle 360,118\rangle$ | 1 | |
| 194 | 27 | $1/5,4/5$ | $3^2,5$ | $4^2,5$ | $\mathcal{A}_6$ | $\langle 360,118\rangle$ | 2 | |
| 195 | 27 | $1/5,4/5$ | $2,4,5$ | $3^3,5$ | $\mathcal{A}_6$ | $\langle 360,118\rangle$ | 6 | |
| 196 | 27 | $1/5,4/5$ | $2,4,5$ | $3,6,10$ | $\mathbb{Z}_2\times\mathcal{A}_6$ | $\langle 720,766\rangle$ | 2 | |
| 197 | 27 | $1/3,1/2^2,2/3$ | $2,4,6$ | $2^8,3,4$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | ? | |
| 198 | 26 | $1/2^6$ | $2^3,4$ | $2^9,4$ | $\mathbb{Z}_2\times D_4$ | $\langle 16,11\rangle$ | 14 | 0 |
| 199 | 26 | $1/2^6$ | $2^3,4$ | $2^6,4^3$ | $\mathbb{Z}_2\times D_4$ | $\langle 16,11\rangle$ | 2 | |
| 200 | 26 | $1/2^6$ | $2,6^2$ | $2^3,3^4$ | $S_3\times S_3$ | $\langle 36,10\rangle$ | 1 | |
| 201 | 26 | $1/2^6$ | $2,3^3,6^2$ | $2^3,3$ | $S_3\times S_3$ | $\langle 36,10\rangle$ | 2 | |
| 202 | 26 | $1/2^6$ | $2^3,4$ | $2^3,4,6$ | $S_3\times D_4$ | $\langle 48,38\rangle$ | 1 | |
| 203 | 26 | $1/2^6$ | $2^3,3,12$ | $2^3,4$ | $S_3\times D_4$ | $\langle 48,38\rangle$ | 1 | |
| 204 | 26 | $1/2^6$ | $2,4,6$ | $2^6,4^3$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | ? | |
| 205 | 26 | $1/2^6$ | $2^2,3^2,4$ | $2^3,4$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | 2 | |
| 206 | 26 | $1/2^6$ | $2^3,4$ | $2^3,4,6$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | 3 | |
| 207 | 26 | $1/2^6$ | $2,4,6$ | $2^9,4$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | ? | |
| 208 | 26 | $1/2^6$ | $2,5^2$ | $2^3,3^4$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 1 | |
| 209 | 26 | $1/2^6$ | $2^3,4$ | $2^3,28$ | $D_4\times D_7$ | $\langle 112,31\rangle$ | 1 | |
| 210 | 26 | $1/2^6$ | $2,3^3,6^2$ | $2,4,5$ | $S_5$ | $\langle 120,34\rangle$ | 1 | |
| 211 | 26 | $1/2^6$ | $2,4,6$ | $2^2,4,10$ | $\mathbb{Z}_2\times S_5$ | $\langle 240,189\rangle$ | 2 | |
| 212 | 26 | $1/2^6$ | $2,6^2$ | $2,7,8$ | SO$(3,7)$ | $\langle 336,208\rangle$ | 2 | |
| 213 | 26 | $1/4,1/2^2,3/4$ | $2^3,4,8$ | $2^3,8$ | $\mathbb{Z}_2\times D_8$ | $\langle 32,39\rangle$ | 2 | |
| 214 | 26 | $1/4,1/2^2,3/4$ | $2,4,5$ | $3^4,4,6$ | $S_5$ | $\langle 120,34\rangle$ | 2 | |

TABLE 21. Minimal product-quotient surfaces of general type with $q=0$, $p_g=3$ and $K^2\in\{28,27,26\}$

| *no.* | $K_S^2$ | Sing$(X)$ | $t_1$ | $t_2$ | $G$ | *Id* | $N$ | deg$(\Phi_S)$ |
|---|---|---|---|---|---|---|---|---|
| 215 | 26 | $1/4, 1/2^2, 3/4$ | $2,4,7$ | $3^3,4$ | PSL$(2,7)$ | $\langle 168,42\rangle$ | 2 | |
| 216 | 26 | $1/4, 1/2^2, 3/4$ | $2,4,7^2$ | $3^2,4$ | PSL$(2,7)$ | $\langle 168,42\rangle$ | 4 | |
| 217 | 26 | $1/4, 1/2^2, 3/4$ | $3^2,4$ | $3^3,4$ | ASL$(2,3)$ | $\langle 216,153\rangle$ | 4 | |
| 218 | 26 | $1/4, 1/2^2, 3/4$ | $2,4,5$ | $3^3,4$ | $\mathcal{A}_6$ | $\langle 360,118\rangle$ | 8 | |
| 219 | 26 | $1/3^2, 2/3^2$ | $2^8,3^2$ | $3,4^2$ | $S_4$ | $\langle 24,12\rangle$ | 1 | |
| 220 | 26 | $1/3^2, 2/3^2$ | $2^2,3^2$ | $2^3,3,4^2$ | $S_4$ | $\langle 24,12\rangle$ | 1 | |
| 221 | 26 | $1/3^2, 2/3^2$ | $2^2,3^2$ | $3,4^4$ | $S_4$ | $\langle 24,12\rangle$ | 2 | |
| 222 | 26 | $1/3^2, 2/3^2$ | $2^4,3$ | $3^2,4^2$ | $S_4$ | $\langle 24,12\rangle$ | 2 | |
| 223 | 26 | $1/3^2, 2/3^2$ | $2^2,3^2$ | $2^4,6^2$ | $\mathbb{Z}_2\times\mathcal{A}_4$ | $\langle 24,13\rangle$ | 2 | |
| 224 | 26 | $1/3^2, 2/3^2$ | $2,6^2$ | $2^8,3^2$ | $\mathbb{Z}_2\times\mathcal{A}_4$ | $\langle 24,13\rangle$ | 1 | |
| 225 | 26 | $1/3^2, 2/3^2$ | $3,9^2$ | $3^2,9^2$ | $\mathbb{Z}_3\times\mathbb{Z}_9$ | $\langle 27,2\rangle$ | 6 | $6^3,7,9,10$ |
| 226 | 26 | $1/3^2, 2/3^2$ | $2^4,3$ | $3,8^2$ | GL$(2,3)$ | $\langle 48,29\rangle$ | 1 | |
| 227 | 26 | $1/3^2, 2/3^2$ | $2^3,3,4^2$ | $3,4^2$ | $\mathcal{A}_4\rtimes\mathbb{Z}_4$ | $\langle 48,30\rangle$ | 3 | |
| 228 | 26 | $1/3^2, 2/3^2$ | $2,4^2,6^2$ | $3,4^2$ | $\mathcal{A}_4\rtimes\mathbb{Z}_4$ | $\langle 48,30\rangle$ | 2 | |
| 229 | 26 | $1/3^2, 2/3^2$ | $3,4^2$ | $3,4^4$ | $\mathcal{A}_4\rtimes\mathbb{Z}_4$ | $\langle 48,30\rangle$ | 2 | |
| 230 | 26 | $1/3^2, 2/3^2$ | $2^3,3$ | $3,4^4$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | 2 | |
| 231 | 26 | $1/3^2, 2/3^2$ | $2^3,3^2$ | $4^2,6$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | 1 | |
| 232 | 26 | $1/3^2, 2/3^2$ | $2,4^2,6^2$ | $2^3,3$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | 3 | |
| 233 | 26 | $1/3^2, 2/3^2$ | $2,4,6$ | $2^8,3^2$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | ? | |
| 234 | 26 | $1/3^2, 2/3^2$ | $2^2,3,4$ | $2^4,3$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | 1 | |
| 235 | 26 | $1/3^2, 2/3^2$ | $2^3,3$ | $2^3,3,4^2$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | 3 | |
| 236 | 26 | $1/3^2, 2/3^2$ | $2^2,3,4$ | $2^2,6^2$ | $\mathbb{Z}_2\times S_4$ | $\langle 48,48\rangle$ | 1 | |
| 237 | 26 | $1/3^2, 2/3^2$ | $2,6^2$ | $2^4,6^2$ | $\mathbb{Z}_2^2\times\mathcal{A}_4$ | $\langle 48,49\rangle$ | 5 | 8 |
| 238 | 26 | $1/3^2, 2/3^2$ | $2^2,3^2$ | $2^3,3^2$ | $\mathbb{Z}_2^4\rtimes\mathbb{Z}_3$ | $\langle 48,50\rangle$ | 2 | |
| 239 | 26 | $1/3^2, 2/3^2$ | $2^3,3^2$ | $3,5^2$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 2 | |
| 240 | 26 | $1/3^2, 2/3^2$ | $2^6,3$ | $3^2,5$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 1 | |
| 241 | 26 | $1/3^2, 2/3^2$ | $2^2,3^2$ | $3,8^2$ | $G(96,64)$ | $\langle 96,64\rangle$ | 2 | |
| 242 | 26 | $1/3^2, 2/3^2$ | $2,6^2$ | $3^2,4^2$ | $\left(\mathbb{Z}_2^2\wr\mathbb{Z}_2\right)\rtimes\mathbb{Z}_3$ | $\langle 96,70\rangle$ | 2 | |
| 243 | 26 | $1/3^2, 2/3^2$ | $2^2,3^2$ | $4,6^2$ | $G(96,72)$ | $\langle 96,72\rangle$ | 2 | |
| 244 | 26 | $1/3^2, 2/3^2$ | $2,6,8$ | $2^2,6^2$ | $\mathbb{Z}_2\times$ GL$(2,3)$ | $\langle 96,189\rangle$ | 1 | |
| 245 | 26 | $1/3^2, 2/3^2$ | $2^2,3,4$ | $4^2,6$ | GL$(2,\mathbb{Z}_4)$ | $\langle 96,195\rangle$ | 2 | |
| 246 | 26 | $1/3^2, 2/3^2$ | $2,4,6$ | $2,4^2,6^2$ | GL$(2,\mathbb{Z}_4)$ | $\langle 96,195\rangle$ | 1 | |
| 247 | 26 | $1/3^2, 2/3^2$ | $2,4,6$ | $3,4^4$ | GL$(2,\mathbb{Z}_4)$ | $\langle 96,195\rangle$ | 1 | |
| 248 | 26 | $1/3^2, 2/3^2$ | $2,4,6$ | $2^3,3,4^2$ | GL$(2,\mathbb{Z}_4)$ | $\langle 96,195\rangle$ | 8 | |
| 249 | 26 | $1/3^2, 2/3^2$ | $2^3,3$ | $3^2,4^2$ | $G(96,227)$ | $\langle 96,227\rangle$ | 4 | |
| 250 | 26 | $1/3^2, 2/3^2$ | $3,4^2$ | $3^2,4^2$ | $\mathbb{Z}_2^2\rtimes S_4$ | $\langle 96,227\rangle$ | 2 | |
| 251 | 26 | $1/3^2, 2/3^2$ | $2^3,3^2$ | $3,4^2$ | $\mathbb{Z}_2^2\rtimes S_4$ | $\langle 96,227\rangle$ | 2 | |
| 252 | 26 | $1/3^2, 2/3^2$ | $2,5,6$ | $3^2,4^2$ | $S_5$ | $\langle 120,34\rangle$ | 2 | |
| 253 | 26 | $1/3^2, 2/3^2$ | $2^2,6^2$ | $3,4^2$ | $S_5$ | $\langle 120,34\rangle$ | 1 | |
| 254 | 26 | $1/3^2, 2/3^2$ | $2^2,3,4$ | $3^2,7$ | PSL$(2,7)$ | $\langle 168,42\rangle$ | 1 | |
| 255 | 26 | $1/3^2, 2/3^2$ | $2,3,8$ | $2,4^2,6^2$ | $G(192,181)$ | $\langle 192,181\rangle$ | 2 | |
| 256 | 26 | $1/3^2, 2/3^2$ | $2,3,8$ | $3,4^4$ | $G(192,181)$ | $\langle 192,181\rangle$ | 1 | |
| 257 | 26 | $1/3^2, 2/3^2$ | $2,6^2$ | $4,6^2$ | $\mathbb{Z}_2\wr\mathcal{A}_4$ | $\langle 192,201\rangle$ | 3 | |
| 258 | 26 | $1/3^2, 2/3^2$ | $2^2,3,4$ | $3,4^2$ | $\mathbb{Z}_2^3\rtimes S_4$ | $\langle 192,1493\rangle$ | 3 | |

TABLE 22. Minimal product-quotient surfaces of general type with $q=0$, $p_g=3$ and $K^2=26$

| *no.* | $K_S^2$ | Sing($X$) | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg($\Phi_S$) |
|---|---|---|---|---|---|---|---|---|
| 259 | 26 | $1/3^2, 2/3^2$ | $2^3, 3$ | $3, 8^2$ | $G(192, 1494)$ | $\langle 192, 1494\rangle$ | 1 | |
| 260 | 26 | $1/3^2, 2/3^2$ | $2, 5, 6$ | $3, 8^2$ | $\mathrm{SL}(2, 5) \rtimes \mathbb{Z}_2$ | $\langle 240, 90\rangle$ | 1 | |
| 261 | 26 | $1/3^2, 2/3^2$ | $2, 12^2$ | $3, 4^2$ | $\mathcal{A}_5 \rtimes \mathbb{Z}_4$ | $\langle 240, 91\rangle$ | 1 | |
| 262 | 26 | $1/3^2, 2/3^2$ | $2, 6^2$ | $4^2, 6$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240, 189\rangle$ | 1 | |
| 263 | 26 | $1/3^2, 2/3^2$ | $3^2, 4$ | $4, 6^2$ | $G(384, 4)$ | $\langle 384, 4\rangle$ | 2 | |
| 264 | 26 | $1/3^2, 2/3^2$ | $2, 3, 11$ | $3, 5^2$ | $\mathrm{PSL}(2, 11)$ | $\langle 660, 13\rangle$ | 6 | |
| 265 | 26 | $1/3^2, 2/3^2$ | $2, 3, 7$ | $3, 13^2$ | $\mathrm{PSL}(2, 13)$ | $\langle 1092, 25\rangle$ | 12 | |
| 266 | 26 | $1/3^2, 2/3^2$ | $2, 3, 7$ | $3, 8^2$ | $G(1344, 814)$ | $\langle 1344, 814\rangle$ | 8 | |
| 267 | 26 | $1/4, 1/2^2, 3/4$ | $3^2, 4$ | $3^2, 4$ | $G(1944, 3875)$ | $\langle 1944, 3875\rangle$ | 2 | |
| 268 | 25 | $1/3, 1/2^4, 2/3$ | $2^9, 3$ | $3, 4^2$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 269 | 25 | $1/3, 1/2^4, 2/3$ | $2, 4, 6$ | $2^6, 3, 4^2$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 270 | 25 | $1/3, 1/2^4, 2/3$ | $2^3, 3$ | $2^4, 4, 6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 4 | |
| 271 | 25 | $1/3, 1/2^4, 2/3$ | $2, 4, 6$ | $2^9, 3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 272 | 25 | $1/3, 1/2^4, 2/3$ | $2, 4^3, 6$ | $2^3, 3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 273 | 25 | $1/3, 1/2^4, 2/3$ | $2^2, 8, 12$ | $2^3, 3$ | $G(96, 193)$ | $\langle 96, 193\rangle$ | 1 | |
| 274 | 25 | $1/3, 1/2^4, 2/3$ | $2, 4, 6$ | $2^4, 4, 6$ | $\mathrm{GL}(2, \mathbb{Z}_4)$ | $\langle 96, 195\rangle$ | 6 | |
| 275 | 25 | $1/3, 1/2^4, 2/3$ | $2, 4, 6$ | $2, 4^3, 6$ | $\mathrm{GL}(2, \mathbb{Z}_4)$ | $\langle 96, 195\rangle$ | 1 | |
| 276 | 25 | $1/3, 1/2^4, 2/3$ | $2, 4, 6$ | $2^2, 3, 5^2$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 277 | 25 | $1/3, 1/2^4, 2/3$ | $2^2, 5, 6$ | $3, 4^2$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 278 | 25 | $1/3, 1/2^4, 2/3$ | $2^2, 5, 6$ | $2^3, 3$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 279 | 25 | $1/3, 1/2^4, 2/3$ | $2, 3, 8$ | $2, 4^3, 6$ | $G(192, 181)$ | $\langle 192, 181\rangle$ | 3 | |
| 280 | 25 | $1/3, 1/2^4, 2/3$ | $2, 4, 6$ | $2^2, 5, 6$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240, 189\rangle$ | 2 | |
| 281 | 25 | $1/3, 1/2^4, 2/3$ | $2, 4, 6$ | $2, 10, 12$ | $\mathbb{Z}_2^2 \rtimes S_5$ | $\langle 480, 951\rangle$ | 2 | |
| 282 | 25 | $1/3, 2/5^2, 2/3$ | $2, 6, 10$ | $2^2, 3, 5$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 283 | 24 | $1/2^8$ | $2^6$ | $2^{10}$ | $\mathbb{Z}_2^2$ | $\langle 4, 2\rangle$ | 1 | 0 |
| 284 | 24 | $1/2^8$ | $2^3, 4^2$ | $2^4, 4^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\langle 8, 2\rangle$ | 1 | 8 |
| 285 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^7, 4^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\langle 8, 2\rangle$ | 1 | 2 |
| 286 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4^4$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\langle 8, 2\rangle$ | 2 | 2, 8 |
| 287 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^{10}$ | $D_4$ | $\langle 8, 3\rangle$ | 1 | |
| 288 | 24 | $1/2^8$ | $2^4, 4^2$ | $2^6$ | $D_4$ | $\langle 8, 3\rangle$ | 1 | |
| 289 | 24 | $1/2^8$ | $2^6$ | $2^7$ | $\mathbb{Z}_2^3$ | $\langle 8, 5\rangle$ | 11 | $4^3, 6^2, 8^3, 12^2, 16$ |
| 290 | 24 | $1/2^8$ | $2^5$ | $2^{10}$ | $\mathbb{Z}_2^3$ | $\langle 8, 5\rangle$ | 14 | $0^4, 4^7, 6, 8^2$ |
| 291 | 24 | $1/2^8$ | $2^2, 6^2$ | $2^7$ | $D_6$ | $\langle 12, 4\rangle$ | 1 | |
| 292 | 24 | $1/2^8$ | $2^2, 3^2, 6^2$ | $2^5$ | $D_6$ | $\langle 12, 4\rangle$ | 1 | |
| 293 | 24 | $1/2^8$ | $2^3, 6$ | $2^{10}$ | $D_6$ | $\langle 12, 4\rangle$ | 1 | |
| 294 | 24 | $1/2^8$ | $2^3, 3, 6$ | $2^6$ | $D_6$ | $\langle 12, 4\rangle$ | 1 | |
| 295 | 24 | $1/2^8$ | $2, 4^3$ | $4^4$ | $\mathbb{Z}_4^2$ | $\langle 16, 2\rangle$ | 1 | 12 |
| 296 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4^2$ | $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$ | $\langle 16, 3\rangle$ | 13 | $8^3$ |
| 297 | 24 | $1/2^8$ | $2^2, 8^2$ | $2^6$ | $D_8$ | $\langle 16, 7\rangle$ | 2 | |
| 298 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4^2$ | $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ | $\langle 16, 10\rangle$ | 10 | $8^4, 12^4, 16^2$ |
| 299 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^7$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 7 | |
| 300 | 24 | $1/2^8$ | $2^4, 4^2$ | $2^5$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 14 | |
| 301 | 24 | $1/2^8$ | $2^3, 4$ | $2^4, 4^4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 1 | |

TABLE 23. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 \in \{26, 25, 24\}$

| *no.* | $K_S^2$ | Sing$(X)$ | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | $\deg(\Phi_S)$ |
|---|---|---|---|---|---|---|---|---|
| 302 | 24 | $1/2^8$ | $2^3, 4^2$ | $2^4, 4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 2 | |
| 303 | 24 | $1/2^8$ | $2^3, 4$ | $2^{10}$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 27 | 0 |
| 304 | 24 | $1/2^8$ | $2^5$ | $2^7$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 4 | 16 |
| 305 | 24 | $1/2^8$ | $2^4, 4$ | $2^6$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 14 | $8^2$ |
| 306 | 24 | $1/2^8$ | $2^3, 4$ | $2^7, 4^2$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16, 11\rangle$ | 9 | |
| 307 | 24 | $1/2^8$ | $2, 4^3$ | $2^6$ | $D_4 \rtimes \mathbb{Z}_2$ | $\langle 16, 13\rangle$ | 1 | |
| 308 | 24 | $1/2^8$ | $2^5$ | $2^7$ | $\mathbb{Z}_2^4$ | $\langle 16, 14\rangle$ | 13 | $8^5, 12^4, 16^4$ |
| 309 | 24 | $1/2^8$ | $2^2, 3^2$ | $3, 6^4$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18, 3\rangle$ | 3 | 0, 6 |
| 310 | 24 | $1/2^8$ | $2, 3^2, 6$ | $2^2, 3^3$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18, 3\rangle$ | 2 | |
| 311 | 24 | $1/2^8$ | $2^4, 3^3$ | $3, 6^2$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18, 3\rangle$ | 1 | |
| 312 | 24 | $1/2^8$ | $2, 3^4, 6$ | $2^2, 3^2$ | $\mathbb{Z}_3 \times S_3$ | $\langle 18, 3\rangle$ | 3 | 6 |
| 313 | 24 | $1/2^8$ | $2^2, 3^2$ | $2^4, 3^3$ | $\mathbb{Z}_3 \rtimes S_3$ | $\langle 18, 4\rangle$ | 2 | |
| 314 | 24 | $1/2^8$ | $2^2, 3^3$ | $2^4, 3$ | $\mathbb{Z}_3 \rtimes S_3$ | $\langle 18, 4\rangle$ | 2 | |
| 315 | 24 | $1/2^8$ | $2^3, 6$ | $2^4, 4^2$ | $\mathbb{Z}_3 \rtimes D_4$ | $\langle 24, 8\rangle$ | 1 | |
| 316 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^3, 3, 6$ | $\mathbb{Z}_3 \rtimes D_4$ | $\langle 24, 8\rangle$ | 1 | |
| 317 | 24 | $1/2^8$ | $2^2, 3^3$ | $2^2, 4^2$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 318 | 24 | $1/2^8$ | $2, 4^3$ | $2^4, 3$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 319 | 24 | $1/2^8$ | $2^{10}$ | $3, 4^2$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 320 | 24 | $1/2^8$ | $2^2, 3^2$ | $2^4, 4^2$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 321 | 24 | $1/2^8$ | $2^5$ | $4^4$ | $S_4$ | $\langle 24, 12\rangle$ | 1 | |
| 322 | 24 | $1/2^8$ | $2^3, 3, 6$ | $2^5$ | $\mathbb{Z}_2^2 \times S_3$ | $\langle 24, 14\rangle$ | 3 | |
| 323 | 24 | $1/2^8$ | $2^3, 6$ | $2^7$ | $\mathbb{Z}_2^2 \times S_3$ | $\langle 24, 14\rangle$ | 11 | |
| 324 | 24 | $1/2^8$ | $2^5$ | $2^6$ | $\mathbb{Z}_2^2 \times S_3$ | $\langle 24, 14\rangle$ | 3 | |
| 325 | 24 | $1/2^8$ | $2^2, 14^2$ | $2^5$ | $D_{14}$ | $\langle 28, 3\rangle$ | 2 | |
| 326 | 24 | $1/2^8$ | $2^3, 14$ | $2^6$ | $D_{14}$ | $\langle 28, 3\rangle$ | 1 | |
| 327 | 24 | $1/2^8$ | $2, 4^3$ | $2^2, 4^2$ | $G(32, 6)$ | $\langle 32, 6\rangle$ | 1 | |
| 328 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^2, 8^2$ | $D_4 \rtimes \mathbb{Z}_4$ | $\langle 32, 9\rangle$ | 6 | |
| 329 | 24 | $1/2^8$ | $2^3, 4^2$ | $4^2, 8$ | $\mathbb{Z}_4 \wr \mathbb{Z}_2$ | $\langle 32, 11\rangle$ | 2 | |
| 330 | 24 | $1/2^8$ | $2, 4^3$ | $2^2, 4^2$ | $\mathbb{Z}_4 \times D_4$ | $\langle 32, 25\rangle$ | 4 | |
| 331 | 24 | $1/2^8$ | $2^4, 4$ | $2^5$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 3 | |
| 332 | 24 | $1/2^8$ | $2^3, 4$ | $2^4, 4^2$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 27 | |
| 333 | 24 | $1/2^8$ | $2, 4^3$ | $2^5$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 1 | |
| 334 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 3 | |
| 335 | 24 | $1/2^8$ | $2^3, 4$ | $2^7$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\langle 32, 27\rangle$ | 10 | |
| 336 | 24 | $1/2^8$ | $2^3, 4$ | $2^4, 4^2$ | $\mathbb{Z}_2^2 \rtimes D_4$ | $\langle 32, 28\rangle$ | 9 | |
| 337 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4$ | $\mathbb{Z}_2^2 \rtimes D_4$ | $\langle 32, 28\rangle$ | 7 | |
| 338 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^4, 4$ | $\mathbb{Z}_4 \rtimes D_4$ | $\langle 32, 34\rangle$ | 3 | |
| 339 | 24 | $1/2^8$ | $2^3, 8$ | $2^6$ | $\mathbb{Z}_2 \times D_8$ | $\langle 32, 39\rangle$ | 5 | |
| 340 | 24 | $1/2^8$ | $2^4, 4$ | $2^5$ | $\mathbb{Z}_2 \times D_8$ | $\langle 32, 39\rangle$ | 1 | |
| 341 | 24 | $1/2^8$ | $2^2, 8^2$ | $2^5$ | $\mathbb{Z}_2 \times D_8$ | $\langle 32, 39\rangle$ | 4 | |
| 342 | 24 | $1/2^8$ | $2^3, 4^2$ | $2^3, 8$ | $\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$ | $\langle 32, 43\rangle$ | 1 | |
| 343 | 24 | $1/2^8$ | $2^2, 8^2$ | $2^5$ | $\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$ | $\langle 32, 43\rangle$ | 1 | |
| 344 | 24 | $1/2^8$ | $2^4, 4$ | $2^5$ | $\mathbb{Z}_2^2 \times D_4$ | $\langle 32, 46\rangle$ | 5 | |
| 345 | 24 | $1/2^8$ | $2^4, 4$ | $2^5$ | $Q_8 \rtimes \mathbb{Z}_2^2$ | $\langle 32, 49\rangle$ | 1 | |

TABLE 24. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 = 24$

| *no.* | $K_S^2$ | Sing(X) | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg($\Phi_S$) |
|---|---|---|---|---|---|---|---|---|
| 346 | 24 | $1/2^8$ | $2, 6^2$ | $2^4, 3^3$ | $S_3 \times S_3$ | $\langle 36, 10\rangle$ | 2 | |
| 347 | 24 | $1/2^8$ | $2^2, 3^2, 6^2$ | $2^3, 3$ | $S_3 \times S_3$ | $\langle 36, 10\rangle$ | 3 | |
| 348 | 24 | $1/2^8$ | $2^2, 3, 6$ | $2^2, 6^2$ | $S_3 \times S_3$ | $\langle 36, 10\rangle$ | 1 | |
| 349 | 24 | $1/2^8$ | $2^3, 3$ | $3, 6^4$ | $S_3 \times S_3$ | $\langle 36, 10\rangle$ | 2 | |
| 350 | 24 | $1/2^8$ | $2^2, 3, 6$ | $2^4, 3$ | $S_3 \times S_3$ | $\langle 36, 10\rangle$ | 1 | |
| 351 | 24 | $1/2^8$ | $2^3, 3, 6$ | $2^3, 6$ | $S_3 \times S_3$ | $\langle 36, 10\rangle$ | 2 | |
| 352 | 24 | $1/2^8$ | $2^2, 6^2$ | $6^3$ | $\mathbb{Z}_6 \times S_3$ | $\langle 36, 12\rangle$ | 1 | |
| 353 | 24 | $1/2^8$ | $2^2, 3, 6$ | $2^2, 6^2$ | $G(36, 13)$ | $\langle 36, 13\rangle$ | 1 | |
| 354 | 24 | $1/2^8$ | $2^2, 3, 6$ | $2^4, 3$ | $G(36, 13)$ | $\langle 36, 13\rangle$ | 1 | |
| 355 | 24 | $1/2^8$ | $2^2, 8^2$ | $2^3, 6$ | $\mathbb{Z}_3 \rtimes D_8$ | $\langle 48, 15\rangle$ | 2 | |
| 356 | 24 | $1/2^8$ | $2^2, 3^2$ | $2^2, 8^2$ | $\mathrm{GL}(2, 3)$ | $\langle 48, 29\rangle$ | 2 | |
| 357 | 24 | $1/2^8$ | $2, 4^4$ | $3, 4^2$ | $\mathcal{A}_4 \rtimes \mathbb{Z}_4$ | $\langle 48, 30\rangle$ | 1 | |
| 358 | 24 | $1/2^8$ | $2^5$ | $2^5$ | $\mathbb{Z}_2 \times D_{12}$ | $\langle 48, 36\rangle$ | 1 | |
| 359 | 24 | $1/2^8$ | $2^3, 4$ | $2^6$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 2 | |
| 360 | 24 | $1/2^8$ | $2^3, 6$ | $2^4, 4$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 5 | |
| 361 | 24 | $1/2^8$ | $2^3, 3, 6$ | $2^3, 4$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 1 | |
| 362 | 24 | $1/2^8$ | $2, 4, 6$ | $2^7, 4^2$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 363 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^2, 4^2$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 364 | 24 | $1/2^8$ | $2, 4, 6$ | $2^{10}$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 365 | 24 | $1/2^8$ | $2^3, 6$ | $2^4, 4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 3 | |
| 366 | 24 | $1/2^8$ | $2, 4, 6$ | $2^4, 4^4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 3 | |
| 367 | 24 | $1/2^8$ | $2^3, 3$ | $2^4, 4^2$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 7 | |
| 368 | 24 | $1/2^8$ | $2^2, 3, 6$ | $2^2, 4^2$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 369 | 24 | $1/2^8$ | $2, 4^4$ | $2^3, 3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 370 | 24 | $1/2^8$ | $2^3, 3, 6$ | $2^3, 4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 371 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^5$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 372 | 24 | $1/2^8$ | $2, 4^3$ | $2^3, 6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 373 | 24 | $1/2^8$ | $2^5$ | $2^5$ | $\mathbb{Z}_2^3 \times S_3$ | $\langle 48, 51\rangle$ | 1 | |
| 374 | 24 | $1/2^8$ | $2, 3^2, 6$ | $2^2, 3^2$ | $\mathrm{He3} \rtimes \mathbb{Z}_2$ | $\langle 54, 5\rangle$ | 1 | |
| 375 | 24 | $1/2^8$ | $2, 3^2, 6$ | $2^2, 3^2$ | $\mathbb{Z}_3^2 \rtimes S_3$ | $\langle 54, 8\rangle$ | 1 | |
| 376 | 24 | $1/2^8$ | $2, 3^2, 6$ | $3, 6^2$ | $S_3 \times \mathbb{Z}_3^2$ | $\langle 54, 12\rangle$ | 9 | 12, (16, 18), (13, 15), 18, 24 |
| 377 | 24 | $1/2^8$ | $2, 3^2, 6$ | $2^2, 3^2$ | $G(54, 13)$ | $\langle 54, 13\rangle$ | 4 | |
| 378 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^3, 14$ | $D_{14} \rtimes \mathbb{Z}_2$ | $\langle 56, 7\rangle$ | 1 | |
| 379 | 24 | $1/2^8$ | $2^3, 14$ | $2^5$ | $\mathbb{Z}_2^2 \times D_7$ | $\langle 56, 12\rangle$ | 3 | |
| 380 | 24 | $1/2^8$ | $2, 5^2$ | $2^4, 3^3$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 1 | |
| 381 | 24 | $1/2^8$ | $2^2, 3^2$ | $2^2, 5^2$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 2 | |
| 382 | 24 | $1/2^8$ | $2^2, 4^2$ | $4^2, 8$ | $G(64, 8)$ | $\langle 64, 8\rangle$ | 1 | |
| 383 | 24 | $1/2^8$ | $2, 4, 8$ | $2, 4^4$ | $G(64, 8)$ | $\langle 64, 8\rangle$ | 4 | |
| 384 | 24 | $1/2^8$ | $2, 4^3$ | $4^3$ | $G(64, 23)$ | $\langle 64, 23\rangle$ | 6 | |
| 385 | 24 | $1/2^8$ | $2^3, 4$ | $2^4, 4$ | $G(64, 73)$ | $\langle 64, 73\rangle$ | 2 | |
| 386 | 24 | $1/2^8$ | $2^3, 4$ | $2^4, 4$ | $G(64, 128)$ | $\langle 64, 128\rangle$ | 1 | |
| 387 | 24 | $1/2^8$ | $2^3, 8$ | $2^5$ | $G(64, 128)$ | $\langle 64, 128\rangle$ | 1 | |
| 388 | 24 | $1/2^8$ | $2^2, 8^2$ | $2^3, 4$ | $G(64, 128)$ | $\langle 64, 128\rangle$ | 1 | |

Table 25. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 = 24$

| *no.* | $K_S^2$ | Sing($X$) | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg($\Phi_S$) |
|---|---|---|---|---|---|---|---|---|
| 389 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^3, 8$ | $G(64, 130)$ | $\langle 64, 130\rangle$ | 1 | |
| 390 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^3, 8$ | $D_4 \rtimes D_4$ | $\langle 64, 134\rangle$ | 1 | |
| 391 | 24 | $1/2^8$ | $2, 4^3$ | $2^3, 4$ | $D_4 \rtimes D_4$ | $\langle 64, 134\rangle$ | 1 | |
| 392 | 24 | $1/2^8$ | $2^3, 4$ | $2^4, 4$ | $\mathbb{Z}_2 \wr \mathbb{Z}_2^2$ | $\langle 64, 138\rangle$ | 5 | |
| 393 | 24 | $1/2^8$ | $2, 4^3$ | $2^3, 4$ | $\mathbb{Z}_2 \wr \mathbb{Z}_2^2$ | $\langle 64, 138\rangle$ | 2 | |
| 394 | 24 | $1/2^8$ | $2^3, 4$ | $2^4, 4$ | $\mathbb{Z}_4 \rtimes D_8$ | $\langle 64, 140\rangle$ | 1 | |
| 395 | 24 | $1/2^8$ | $2^2, 8^2$ | $2^3, 4$ | $\mathbb{Z}_4 \rtimes D_8$ | $\langle 64, 140\rangle$ | 1 | |
| 396 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^3, 8$ | $\mathbb{Z}_2^2 \rtimes D_8$ | $\langle 64, 147\rangle$ | 1 | |
| 397 | 24 | $1/2^8$ | $2^2, 4^2$ | $2^3, 8$ | $G(64, 150)$ | $\langle 64, 150\rangle$ | 1 | |
| 398 | 24 | $1/2^8$ | $2, 6, 12$ | $2^4, 3$ | $\mathbb{Z}_3^2 \rtimes D_4$ | $\langle 72, 23\rangle$ | 1 | |
| 399 | 24 | $1/2^8$ | $2, 4, 6$ | $2^2, 3^2, 6^2$ | $S_3 \wr \mathbb{Z}_2$ | $\langle 72, 40\rangle$ | 1 | |
| 400 | 24 | $1/2^8$ | $2^2, 3, 6$ | $2^2, 3^2$ | $\mathbb{Z}_3 \rtimes S_4$ | $\langle 72, 43\rangle$ | 1 | |
| 401 | 24 | $1/2^8$ | $2^2, 3^2$ | $2^2, 4^2$ | $\mathbb{Z}_3 \rtimes S_4$ | $\langle 72, 43\rangle$ | 1 | |
| 402 | 24 | $1/2^8$ | $2^2, 3^2$ | $6^3$ | $S_3 \times \mathcal{A}_4$ | $\langle 72, 44\rangle$ | 1 | |
| 403 | 24 | $1/2^8$ | $2^3, 6$ | $2^5$ | $\mathbb{Z}_2 \times S_3 \times S_3$ | $\langle 72, 46\rangle$ | 1 | |
| 404 | 24 | $1/2^8$ | $2^3, 6$ | $2^3, 14$ | $S_3 \times D_7$ | $\langle 84, 8\rangle$ | 1 | |
| 405 | 24 | $1/2^8$ | $2^3, 4$ | $2^5$ | $\mathbb{Z}_2^2 \rtimes D_{12}$ | $\langle 96, 89\rangle$ | 1 | |
| 406 | 24 | $1/2^8$ | $2^3, 6$ | $2^3, 8$ | $S_3 \times D_8$ | $\langle 96, 117\rangle$ | 1 | |
| 407 | 24 | $1/2^8$ | $2, 4, 12$ | $2, 4^3$ | $\mathbb{Z}_4 \times S_4$ | $\langle 96, 186\rangle$ | 3 | |
| 408 | 24 | $1/2^8$ | $2, 4, 12$ | $2^4, 4$ | $\mathbb{Z}_4 \rtimes S_4$ | $\langle 96, 187\rangle$ | 2 | |
| 409 | 24 | $1/2^8$ | $2^2, 8^2$ | $2^3, 3$ | $G(96, 193)$ | $\langle 96, 193\rangle$ | 2 | |
| 410 | 24 | $1/2^8$ | $2, 4, 6$ | $2^4, 4^2$ | $\mathrm{GL}(2, \mathbb{Z}_4)$ | $\langle 96, 195\rangle$ | 6 | |
| 411 | 24 | $1/2^8$ | $2, 4, 6$ | $2, 4^4$ | $\mathrm{GL}(2, \mathbb{Z}_4)$ | $\langle 96, 195\rangle$ | 3 | |
| 412 | 24 | $1/2^8$ | $2^4, 4$ | $3, 4^2$ | $\mathbb{Z}_2^2 \rtimes S_4$ | $\langle 96, 227\rangle$ | 1 | |
| 413 | 24 | $1/2^8$ | $2, 4^3$ | $2^3, 3$ | $\mathbb{Z}_2^2 \rtimes S_4$ | $\langle 96, 227\rangle$ | 2 | |
| 414 | 24 | $1/2^8$ | $2, 6^2$ | $2^4, 3$ | $G(108, 17)$ | $\langle 108, 17\rangle$ | 1 | |
| 415 | 24 | $1/2^8$ | $2^3, 4$ | $2^3, 14$ | $D_4 \times D_7$ | $\langle 112, 31\rangle$ | 1 | |
| 416 | 24 | $1/2^8$ | $2, 4, 5$ | $3, 6^4$ | $S_5$ | $\langle 120, 34\rangle$ | 2 | |
| 417 | 24 | $1/2^8$ | $2^3, 5$ | $3, 6^2$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 418 | 24 | $1/2^8$ | $2, 4^3$ | $2, 5, 6$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 419 | 24 | $1/2^8$ | $2, 6^2$ | $2^2, 5^2$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 420 | 24 | $1/2^8$ | $2^3, 6$ | $4^2, 5$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 421 | 24 | $1/2^8$ | $2, 3^4, 6$ | $2, 4, 5$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 422 | 24 | $1/2^8$ | $2, 4, 5$ | $2^2, 3^2, 6^2$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 423 | 24 | $1/2^8$ | $2, 5, 6$ | $2^4, 4$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 424 | 24 | $1/2^8$ | $2, 5, 10$ | $2^2, 3, 6$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 425 | 24 | $1/2^8$ | $2, 5, 10$ | $2^5$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 426 | 24 | $1/2^8$ | $2, 10^2$ | $2^3, 6$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 427 | 24 | $1/2^8$ | $2, 3, 10$ | $2^7$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 428 | 24 | $1/2^8$ | $2^2, 5^2$ | $2^3, 3$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 429 | 24 | $1/2^8$ | $2, 4, 8$ | $2, 4^3$ | $G(128, 75)$ | $\langle 128, 75\rangle$ | 4 | |
| 430 | 24 | $1/2^8$ | $2^3, 4$ | $2^3, 8$ | $G(128, 327)$ | $\langle 128, 327\rangle$ | 1 | |
| 431 | 24 | $1/2^8$ | $2^3, 4$ | $2^3, 8$ | $G(128, 928)$ | $\langle 128, 928\rangle$ | 1 | |
| 432 | 24 | $1/2^8$ | $2, 4, 5$ | $2, 4^4$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160, 234\rangle$ | 5 | |
| 433 | 24 | $1/2^8$ | $2, 4, 5$ | $2^4, 4^2$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160, 234\rangle$ | ? | |

TABLE 26. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 = 24$

| *no.* | $K_S^2$ | Sing$(X)$ | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg$(\Phi_S)$ |
|---|---|---|---|---|---|---|---|---|
| 434 | 24 | $1/2^8$ | $2,6,9$ | $3,6^2$ | $\mathbb{Z}_3 \wr S_3$ | $\langle 162,10\rangle$ | 4 | |
| 435 | 24 | $1/2^8$ | $2,7^2$ | $2^2,3^2$ | PSL$(2,7)$ | $\langle 168,42\rangle$ | 1 | |
| 436 | 24 | $1/2^8$ | $2^4,7$ | $3^2,4$ | PSL$(2,7)$ | $\langle 168,42\rangle$ | 2 | |
| 437 | 24 | $1/2^8$ | $2,3,8$ | $2,4^4$ | $G(192,181)$ | $\langle 192,181\rangle$ | 1 | |
| 438 | 24 | $1/2^8$ | $2,4,6$ | $2^4,4$ | $G(192,955)$ | $\langle 192,955\rangle$ | 5 | |
| 439 | 24 | $1/2^8$ | $2,4,6$ | $2,4^3$ | $G(192,955)$ | $\langle 192,955\rangle$ | 1 | |
| 440 | 24 | $1/2^8$ | $2,4,6$ | $2^2,6^2$ | $G(216,87)$ | $\langle 216,87\rangle$ | 1 | |
| 441 | 24 | $1/2^8$ | $2,4,10$ | $2^3,6$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240,189\rangle$ | 1 | |
| 442 | 24 | $1/2^8$ | $2,10^2$ | $2^3,3$ | $\mathbb{Z}_2^2 \times \mathcal{A}_5$ | $\langle 240,190\rangle$ | 1 | |
| 443 | 24 | $1/2^8$ | $2,4,5$ | $2^2,8^2$ | $G(320,1582)$ | $\langle 320,1582\rangle$ | 5 | |
| 444 | 24 | $1/2^8$ | $2,4,10$ | $2^3,4$ | $G(320,1636)$ | $\langle 320,1636\rangle$ | 2 | |
| 445 | 24 | $1/2^8$ | $2,5^2$ | $2^2,3^2$ | $\mathcal{A}_6$ | $\langle 360,118\rangle$ | 1 | |
| 446 | 24 | $1/2^8$ | $2,3,10$ | $2^2,3,6$ | $S_3 \times \mathcal{A}_5$ | $\langle 360,121\rangle$ | 1 | |
| 447 | 24 | $1/2^8$ | $2,4,6$ | $2^3,8$ | $G(384,5602)$ | $\langle 384,5602\rangle$ | 3 | |
| 448 | 24 | $1/2^8$ | $2,3,8$ | $2,3^2,6$ | AGL$(2,3)$ | $\langle 432,734\rangle$ | 2 | |
| 449 | 24 | $1/2^8$ | $2,3,8$ | $2,6,21$ | $G(1008,881)$ | $\langle 1008,881\rangle$ | 4 | |
| 450 | 24 | $2/5^2,1/2^4$ | $2,4,5$ | $2^4,4,5$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160,234\rangle$ | ? | |
| 451 | 24 | $2/5^2,1/2^4$ | $2,4,5$ | $2,4^3,5$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160,234\rangle$ | 4 | |
| 452 | 24 | $2/5^2,1/2^4$ | $2,4,5$ | $2^2,8,10$ | $G(320,1582)$ | $\langle 320,1582\rangle$ | 4 | |
| 453 | 24 | $2/5^4$ | $2^4,5^2$ | $3^2,5$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 1 | |
| 454 | 24 | $2/5^4$ | $2^4,5$ | $3,5^2$ | $\mathcal{A}_5$ | $\langle 60,5\rangle$ | 1 | |
| 455 | 24 | $2/5^4$ | $2,5^2$ | $2^4,5^2$ | $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$ | $\langle 80,49\rangle$ | 5 | |
| 456 | 24 | $2/5^4$ | $3,15^2$ | $3^2,5$ | $\mathbb{Z}_3 \times \mathcal{A}_5$ | $\langle 180,19\rangle$ | 1 | |
| 457 | 24 | $2/5^4$ | $2,4,5$ | $2,4^2,5^2$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160,234\rangle$ | 6 | |
| 458 | 24 | $2/5^4$ | $2,5^2$ | $2,5^2$ | $G(1280,\cdot)$ | $\langle 1280,1116310\rangle$ | 2 | |
| 459 | 24 | $1/4^2,3/4^2$ | $2^3,4$ | $2^9,4$ | $\mathbb{Z}_2 \times D_4$ | $\langle 16,11\rangle$ | 6 | 0 |
| 460 | 24 | $1/4^2,3/4^2$ | $2^9,4$ | $3,4^2$ | $S_4$ | $\langle 24,12\rangle$ | 1 | |
| 461 | 24 | $1/4^2,3/4^2$ | $2^3,4$ | $3^4,4^2$ | $S_4$ | $\langle 24,12\rangle$ | 2 | |
| 462 | 24 | $1/4^2,3/4^2$ | $3,4^2$ | $3^4,4^2$ | $G(36,9)$ | $\langle 36,9\rangle$ | 1 | |
| 463 | 24 | $1/4^2,3/4^2$ | $2,4,6$ | $2^9,4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48,48\rangle$ | ? | |
| 464 | 24 | $1/4^2,3/4^2$ | $2,4,5$ | $3^4,4^2$ | $S_5$ | $\langle 120,34\rangle$ | 2 | |
| 465 | 24 | $1/4^2,3/4^2$ | $3,4^2$ | $4,7^2$ | PSL$(2,7)$ | $\langle 168,42\rangle$ | 2 | |
| 466 | 24 | $1/4^2,3/4^2$ | $2,4,6$ | $2^2,4,10$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240,189\rangle$ | 1 | |
| 467 | 24 | $1/4,1/2^4,3/4$ | $2,3^3,4,6$ | $2,4,6$ | $S_3 \wr \mathbb{Z}_2$ | $\langle 72,40\rangle$ | 1 | |
| 468 | 24 | $1/4,1/2^4,3/4$ | $2,3^3,4,6$ | $2,4,5$ | $S_5$ | $\langle 120,34\rangle$ | 1 | |
| 469 | 24 | $1/4,1/2^4,3/4$ | $2,4,14$ | $2,4,14$ | $D_7 \wr \mathbb{Z}_2$ | $\langle 392,37\rangle$ | 2 | |
| 470 | 24 | $1/3^2,1/2^2,2/3^2$ | $2,4,6$ | $2^6,3^2,4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48,48\rangle$ | ? | |
| 471 | 24 | $1/3^2,1/2^2,2/3^2$ | $2,3,7^2$ | $3^2,4$ | PSL$(2,7)$ | $\langle 168,42\rangle$ | 4 | |
| 472 | 24 | $3/10^2,1/2^2$ | $2,3,10$ | $2,8,10$ | $G(720,764)$ | $\langle 720,764\rangle$ | 2 | |
| 473 | 24 | $3/10^2,1/2^2$ | $2,3,10$ | $2,4,10$ | $G(1320,133)$ | $\langle 1320,133\rangle$ | 2 | |
| 474 | 24 | $3/8^2,1/2,3/4$ | $2,3,8$ | $2,4^3,8$ | $G(192,181)$ | $\langle 192,181\rangle$ | 2 | |
| 475 | 23 | $1/3^3,2/3^3$ | $3^4$ | $3^6$ | $\mathbb{Z}_3^2$ | $\langle 9,2\rangle$ | 6 | $6^5,9$ |
| 476 | 23 | $1/3^3,2/3^3$ | $2^6,3^3$ | $3,4^2$ | $S_4$ | $\langle 24,12\rangle$ | 1 | |
| 477 | 23 | $1/3^3,2/3^3$ | $2^2,3^2$ | $2^4,3,6$ | $\mathbb{Z}_2 \times \mathcal{A}_4$ | $\langle 24,13\rangle$ | 2 | 8 |
| 478 | 23 | $1/3^3,2/3^3$ | $2^2,3^2$ | $2^2,6^3$ | $\mathbb{Z}_2 \times \mathcal{A}_4$ | $\langle 24,13\rangle$ | 1 | |

TABLE 27. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 \in \{24, 23\}$

| *no.* | $K_S^2$ | Sing$(X)$ | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg$(\Phi_S)$ |
|---|---|---|---|---|---|---|---|---|
| 479 | 23 | $1/3^3, 2/3^3$ | $2, 6^2$ | $2^6, 3^3$ | $\mathbb{Z}_2 \times \mathcal{A}_4$ | $\langle 24, 13\rangle$ | 1 | |
| 480 | 23 | $1/3^3, 2/3^3$ | $3^4$ | $3^4$ | He3 | $\langle 27, 3\rangle$ | 5 | |
| 481 | 23 | $1/3^3, 2/3^3$ | $2, 3^3$ | $3^4$ | $\mathbb{Z}_3 \times \mathcal{A}_4$ | $\langle 36, 11\rangle$ | 4 | |
| 482 | 23 | $1/3^3, 2/3^3$ | $3^2, 6$ | $3^6$ | $\mathbb{Z}_3 \times \mathcal{A}_4$ | $\langle 36, 11\rangle$ | 6 | |
| 483 | 23 | $1/3^3, 2/3^3$ | $2, 3, 4^2, 6$ | $3, 4^2$ | $\mathcal{A}_4 \rtimes \mathbb{Z}_4$ | $\langle 48, 30\rangle$ | 2 | |
| 484 | 23 | $1/3^3, 2/3^3$ | $2, 3, 4^2, 6$ | $2^3, 3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 3 | |
| 485 | 23 | $1/3^3, 2/3^3$ | $2, 4, 6$ | $2^6, 3^3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 486 | 23 | $1/3^3, 2/3^3$ | $2, 6^2$ | $2^4, 3, 6$ | $\mathbb{Z}_2^2 \times \mathcal{A}_4$ | $\langle 48, 49\rangle$ | 6 | 8 |
| 487 | 23 | $1/3^3, 2/3^3$ | $2, 6^2$ | $2^2, 6^3$ | $\mathbb{Z}_2^2 \times \mathcal{A}_4$ | $\langle 48, 49\rangle$ | 1 | |
| 488 | 23 | $1/3^3, 2/3^3$ | $3^2, 21$ | $3^4$ | $(\mathbb{Z}_3 \times \mathbb{Z}_7) \rtimes \mathbb{Z}_3$ | $\langle 63, 3\rangle$ | 8 | |
| 489 | 23 | $1/3^3, 2/3^3$ | $2^2, 3^2$ | $3, 12^2$ | $\mathbb{Z}_3 \times S_4$ | $\langle 72, 42\rangle$ | 1 | |
| 490 | 23 | $1/3^3, 2/3^3$ | $2^2, 3^2$ | $6^3$ | $S_3 \times \mathcal{A}_4$ | $\langle 72, 44\rangle$ | 1 | |
| 491 | 23 | $1/3^3, 2/3^3$ | $2^2, 3, 6$ | $2^2, 3^2$ | $S_3 \times \mathcal{A}_4$ | $\langle 72, 44\rangle$ | 3 | |
| 492 | 23 | $1/3^3, 2/3^3$ | $3^2, 9$ | $3^4$ | $\mathbb{Z}_3 \wr \mathbb{Z}_3$ | $\langle 81, 7\rangle$ | 4 | |
| 493 | 23 | $1/3^3, 2/3^3$ | $3^2, 9$ | $3^4$ | He3 $\rtimes \mathbb{Z}_3$ | $\langle 81, 9\rangle$ | 8 | |
| 494 | 23 | $1/3^3, 2/3^3$ | $2^2, 6^3$ | $3^2, 4$ | $G(96, 3)$ | $\langle 96, 3\rangle$ | 3 | |
| 495 | 23 | $1/3^3, 2/3^3$ | $2, 3, 4^2, 6$ | $2, 4, 6$ | $\mathrm{GL}(2, \mathbb{Z}_4)$ | $\langle 96, 195\rangle$ | 1 | |
| 496 | 23 | $1/3^3, 2/3^3$ | $3^2, 6$ | $3^4$ | $\mathbb{Z}_6^2 \rtimes \mathbb{Z}_3$ | $\langle 108, 22\rangle$ | 12 | |
| 497 | 23 | $1/3^3, 2/3^3$ | $2, 3^3$ | $3^2, 6$ | $\mathcal{A}_4 \times \mathcal{A}_4$ | $\langle 144, 184\rangle$ | 2 | |
| 498 | 23 | $1/3^3, 2/3^3$ | $2, 6^2$ | $2^2, 3, 6$ | $\mathbb{Z}_2 \times S_3 \times \mathcal{A}_4$ | $\langle 144, 190\rangle$ | 2 | |
| 499 | 23 | $1/3^3, 2/3^3$ | $2^2, 3, 6$ | $3^2, 5$ | $\mathbb{Z}_3 \times \mathcal{A}_5$ | $\langle 180, 19\rangle$ | 1 | |
| 500 | 23 | $1/3^3, 2/3^3$ | $2, 3, 15$ | $3^4$ | $\mathbb{Z}_3 \times \mathcal{A}_5$ | $\langle 180, 19\rangle$ | 1 | |
| 501 | 23 | $1/3^3, 2/3^3$ | $2, 3, 4^2, 6$ | $2, 3, 8$ | $G(192, 181)$ | $\langle 192, 181\rangle$ | 2 | |
| 502 | 23 | $1/3^3, 2/3^3$ | $3^2, 4$ | $3^4$ | $\mathrm{ASL}(2, 3)$ | $\langle 216, 153\rangle$ | 2 | |
| 503 | 23 | $1/3^3, 2/3^3$ | $3^2, 9$ | $3^2, 9$ | $(\mathrm{He3} \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ | $\langle 243, 26\rangle$ | 7 | |
| 504 | 23 | $1/3^3, 2/3^3$ | $3^2, 9$ | $3^2, 9$ | $G(243, 28)$ | $\langle 243, 28\rangle$ | 18 | |
| 505 | 23 | $1/3^3, 2/3^3$ | $3^2, 6$ | $3^2, 21$ | $(\mathcal{A}_4 \times \mathbb{Z}_7) \rtimes \mathbb{Z}_3$ | $\langle 252, 27\rangle$ | 4 | |
| 506 | 23 | $1/3^3, 2/3^3$ | $2, 3, 12$ | $2^2, 3, 6$ | $\mathcal{A}_4 \times S_4$ | $\langle 288, 1024\rangle$ | 2 | |
| 507 | 23 | $1/3^3, 2/3^3$ | $3^2, 6$ | $3^2, 9$ | $\left(\mathbb{Z}_2^2 \times \mathrm{He3}\right) \rtimes \mathbb{Z}_3$ | $\langle 324, 54\rangle$ | 9 | |
| 508 | 23 | $1/3^3, 2/3^3$ | $3^2, 6$ | $3^2, 6$ | $(\mathbb{Z}_3 \times \mathcal{A}_4) \rtimes \mathcal{A}_4$ | $\langle 432, 526\rangle$ | 6 | |
| 509 | 23 | $1/3^3, 2/3^3$ | $3^2, 4$ | $3^2, 21$ | $\mathbb{Z}_3 \times \mathrm{PSL}(2, 7)$ | $\langle 504, 157\rangle$ | 4 | |
| 510 | 23 | $1/3^3, 2/3^3$ | $3^2, 4$ | $3^2, 6$ | $G(864, 2666)$ | $\langle 864, 2666\rangle$ | 8 | |
| 511 | 23 | $1/3^3, 2/3^3$ | $2, 3, 7$ | $3, 6, 8$ | $G(1344, 814)$ | $\langle 1344, 814\rangle$ | 16 | |
| 512 | 23 | $3/8, 1/2^4, 5/8$ | $2^3, 16$ | $2^3, 16$ | $\mathbb{Z}_2 \times D_{16}$ | $\langle 64, 186\rangle$ | 2 | |
| 513 | 23 | $1/3, 1/2^6, 2/3$ | $2^2, 4, 6$ | $2^3, 12$ | $\mathbb{Z}_2 \times D_{12}$ | $\langle 48, 36\rangle$ | 1 | |
| 514 | 23 | $1/3, 1/2^6, 2/3$ | $2^2, 4, 6$ | $2^3, 12$ | $S_3 \times D_4$ | $\langle 48, 38\rangle$ | 1 | |
| 515 | 23 | $1/3, 1/2^6, 2/3$ | $2^2, 3, 4$ | $2^2, 4, 6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 516 | 23 | $1/3, 1/2^6, 2/3$ | $2, 4, 6$ | $2^7, 3, 4$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 517 | 23 | $1/3, 1/2^6, 2/3$ | $2, 4, 6$ | $2^4, 3, 4^3$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 518 | 23 | $1/3, 1/2^6, 2/3$ | $2^2, 3, 4$ | $3, 4, 8$ | $G(96, 64)$ | $\langle 96, 64\rangle$ | 1 | |
| 519 | 23 | $1/3, 1/2^6, 2/3$ | $2, 6, 8$ | $2^2, 4, 6$ | $\mathbb{Z}_2 \times \mathrm{GL}(2, 3)$ | $\langle 96, 189\rangle$ | 1 | |
| 520 | 23 | $1/3, 1/2^6, 2/3$ | $2, 6, 18$ | $2^3, 9$ | $S_3 \times D_9$ | $\langle 108, 16\rangle$ | 3 | |
| 521 | 23 | $1/3, 1/2^6, 2/3$ | $2, 4, 6$ | $2^2, 4, 6$ | $\mathbb{Z}_2 \times S_5$ | $\langle 240, 189\rangle$ | 3 | |
| 522 | 23 | $1/3, 1/2^6, 2/3$ | $2, 4, 6$ | $2, 6, 8$ | $\mathbb{Z}_2 \times \mathrm{SO}(3, 7)$ | $\langle 672, 1254\rangle$ | 4 | |

TABLE 28. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 = 23$

| *no.* | $K_S^2$ | Sing($X$) | $t_1$ | $t_2$ | $G$ | $Id$ | $N$ | deg($\Phi_S$) |
|---|---|---|---|---|---|---|---|---|
| 523 | 25 | $1/7, 2/7^2$ | $2,4,7$ | $3^3,7$ | PSL(2, 7) | $\langle 168, 42\rangle$ | 2 | |
| 524 | 25 | $1/7, 2/7^2$ | $2,4,7$ | $3,6,14$ | $\mathbb{Z}_2 \times$ PSL(2, 7) | $\langle 336, 209\rangle$ | 1 | |
| 525 | 25 | $1/7, 2/7^2$ | $2,3,7$ | $4,7,8$ | $G(1344, 814)$ | $\langle 1344, 814\rangle$ | 8 | |
| 526 | 24 | $1/5, 1/3, 2/3, 4/5$ | $2,6,10$ | $2^2,3,5$ | $\mathbb{Z}_2 \times \mathcal{A}_5$ | $\langle 120, 35\rangle$ | 1 | |
| 527 | 24 | $1/6^2, 1/2^2, 2/3$ | $2,4,6$ | $4^4,6$ | GL(2, $\mathbb{Z}_4$) | $\langle 96, 195\rangle$ | 2 | |
| 528 | 24 | $1/6^2, 1/2^2, 2/3$ | $2,4,6$ | $2^3,4^2,6$ | GL(2, $\mathbb{Z}_4$) | $\langle 96, 195\rangle$ | 14 | |
| 529 | 24 | $1/6^2, 1/2^2, 2/3$ | $2,5,6$ | $4,6,8$ | SL(2, 5) $\rtimes \mathbb{Z}_2$ | $\langle 240, 90\rangle$ | 2 | |
| 530 | 24 | $1/6^2, 1/2^2, 2/3$ | $2,4,6$ | $4,6,8$ | $G(384, 5604)$ | $\langle 384, 5604\rangle$ | 4 | |
| 531 | 24 | $1/6^2, 1/2^2, 2/3$ | $2,4,6$ | $4,6,8$ | $G(384, 5677)$ | $\langle 384, 5677\rangle$ | 4 | |
| 532 | 24 | $1/4^4, 1/2^2$ | $2,4^2,8$ | $2^2,4^2$ | $\mathbb{Z}_4 \wr \mathbb{Z}_2$ | $\langle 32, 11\rangle$ | 1 | |
| 533 | 24 | $1/4^4, 1/2^2$ | $2^3,4$ | $2^3,4^3$ | $\mathbb{Z}_2^2 \rtimes D_4$ | $\langle 32, 28\rangle$ | 4 | |
| 534 | 24 | $1/4^4, 1/2^2$ | $2,4,8$ | $2^3,4^3$ | $\mathbb{Z}_2 \wr \mathbb{Z}_4$ | $\langle 64, 32\rangle$ | 4 | |
| 535 | 24 | $1/4^4, 1/2^2$ | $2,4,8$ | $2,4^2,8$ | $G(128, 136)$ | $\langle 128, 136\rangle$ | 1 | |
| 536 | 24 | $1/4^4, 1/2^2$ | $2,3,8$ | $4^5$ | $G(192, 181)$ | $\langle 192, 181\rangle$ | 1 | |
| 537 | 24 | $1/8^2, 1/4, 1/2$ | $2,3,8$ | $2^2,4^3,8$ | $G(192, 181)$ | $\langle 192, 181\rangle$ | 3 | |
| 538 | 24 | $1/6, 1/2^2, 5/6$ | $2,4,6$ | $2^9,6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | ? | |
| 539 | 24 | $1/6, 1/2^2, 5/6$ | $2,4^2,6$ | $2^3,6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 1 | |
| 540 | 24 | $1/6, 1/2^2, 5/6$ | $2^4,6$ | $4^2,6$ | $\mathbb{Z}_2 \times S_4$ | $\langle 48, 48\rangle$ | 2 | |
| 541 | 24 | $1/6, 1/2^2, 5/6$ | $2,4,6$ | $2,4^2,6$ | $G(192, 955)$ | $\langle 192, 955\rangle$ | 4 | |
| 542 | 24 | $1/6, 1/2^2, 5/6$ | $2,6,8$ | $2^3,6$ | $G(192, 956)$ | $\langle 192, 956\rangle$ | 1 | |
| 543 | 24 | $1/6, 1/2^2, 5/6$ | $2,6,7$ | $2,6,8$ | SO(3, 7) | $\langle 336, 208\rangle$ | 2 | |
| 544 | 24 | $1/6, 1/2^2, 5/6$ | $2,4,6$ | $2,6,8$ | $G(768, 1086051)$ | $\langle 768, 1086051\rangle$ | 2 | |
| 545 | 24 | $1/4, 1/2, 5/8^2$ | $2,3,8$ | $2,4^3,8$ | $G(192, 181)$ | $\langle 192, 181\rangle$ | 2 | |
| 546 | 23 | $1/5^5$ | $2,5^2$ | $5^2,15$ | $\mathbb{Z}_5 \times \mathcal{A}_5$ | $\langle 300, 22\rangle$ | 2 | |
| 547 | 23 | $1/5, 2/5^2, 4/5$ | $2^4,5^2$ | $3^2,5$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 1 | |
| 548 | 23 | $1/5, 2/5^2, 4/5$ | $2^4,5$ | $3,5^2$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 1 | |
| 549 | 23 | $1/5, 2/5^2, 4/5$ | $2,5^2$ | $3^5,5$ | $\mathcal{A}_5$ | $\langle 60, 5\rangle$ | 2 | |
| 550 | 23 | $1/5, 2/5^2, 4/5$ | $2,4^2,5$ | $2,5,6$ | $S_5$ | $\langle 120, 34\rangle$ | 1 | |
| 551 | 23 | $1/5, 2/5^2, 4/5$ | $2,4,5$ | $2,4^2,5^2$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160, 234\rangle$ | 6 | |
| 552 | 23 | $1/5, 2/5^2, 4/5$ | $3,15^2$ | $3^2,5$ | $\mathbb{Z}_3 \times \mathcal{A}_5$ | $\langle 180, 19\rangle$ | 1 | |
| 553 | 23 | $1/5, 1/2^4, 4/5$ | $2,4,5$ | $2^4,4,5$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160, 234\rangle$ | ? | |
| 554 | 23 | $1/5, 1/2^4, 4/5$ | $2,4,5$ | $2,4^3,5$ | $\mathbb{Z}_2^4 \rtimes D_5$ | $\langle 160, 234\rangle$ | 4 | |
| 555 | 23 | $1/5, 1/2^4, 4/5$ | $2,4,5$ | $2^2,8,10$ | $G(320, 1582)$ | $\langle 320, 1582\rangle$ | 4 | |

TABLE 29. Remaining product-quotient surfaces of general type with $q = 0$, $p_g = 3$ and $K^2 \in \{23, \ldots, 32\}$ whose minimality is not established

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